TWO APPROXIMATIONS OF A SINE FUNCTION - A COMPARISON

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Introduction

This paper discusses a comparative study of trigonometric sine functions, obtained separately by Bhaskara I (7 th C.), and Bhaskara II (12 th C.) and other mathematicians.

The *Mahābhāskarīya* of Bhāskara I contains a simple but elegant algebraic formula for approximating the trigonometric sine function as,

$$\text{Sin x} \approx \frac{4\text{x}(180-\text{x})}{(40500-\text{x}(180-\text{x}))}$$

where the angular arc *x* is in degrees.

An equivalent form of this formula has been given by Bhāskara II and by almost all subsequent Indian mathematicians. The accuracy of this rule is discussed and comparison with actual values of sine function is given, and also depicted in a graph. Proofs of these results are given.

No one is aware of the process by which Bhaskara I , Bhaskara II and other mathematicians arrived at these formulae, without any tools. But these results reflect the high standard of Indian Mathematicians in ancient period.

The Rule

The rule stating the approximate expression for the trigonometric Sine function is given by Bhaskara I in his first work *Mahābhāskarīya*. The relevant Sanskrit text is,

मख्यादि रहितं कर्म वक्ष्यते तत्समासतः ।
चक्रार्धांशक समूहाद्विशोध्या ये भुजांशका ।। १७ ।।
तत्छेष गुणिता द्विष्ठाः शोध्याः खाभ्रेषुखाब्धितः ।
चतुर्थांशेन शेषस्य द्विष्ठमन्त्य फलं हतम् ।। १८ ।।
बाहु कोट्योः फलं कृत्सनं क्रमोत्क्रम गुणस्य वा ।
लभ्यते चन्द्रतीक्ष्णांश्वोस्ताराणां वापि तत्त्वतः ।। १९ ।।

(*Mahābhāskarīya*, VII, 17-19)

Meaning of Stanza (17-19) - *(Now) I breafly state the rule (for finding the bhujaphala and the koṭiphala, etc.) without use of Rsine-differences, 225, etc. Subtract the degrees of the bhuja (or* *koṭi) from the degrees of half a circle (i.e. 180 degrees). Then multiply the remainder by degrees of the bhuja (or koṭi) and put down the result at two places. At one place subtract the result from 40500. By one-fourth of the remainder (thus obtained) divide the result at the other place as multiplied by the antyaphala (i.e. the epicyclic radius). Thus is obtained the direct the entire bāhuphala (or koṭiphala) for the sun, moon or the star-planets. So also are obtained the direct and inverse Rsine.*

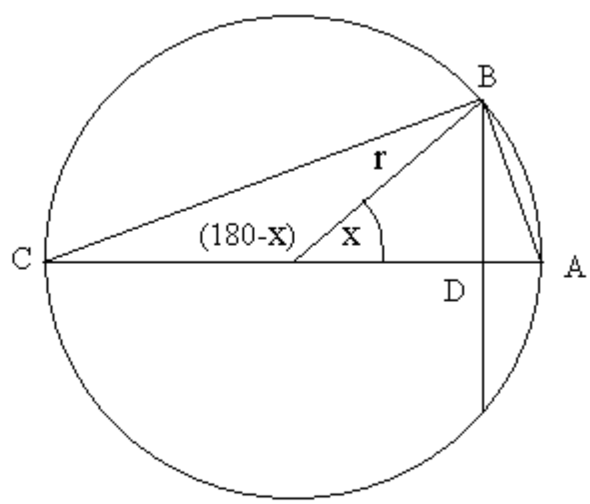
1. The Rules given by Bhaskara I in Mahabhaskariya

The following four results are given by Bhaskara I

(i) To prove that, for the angular arc x is in degrees, $0 \leq x \leq 180^0$.

$$\text{Sin x} = \frac{4x(180-x)}{(40500-x(180-x))} \qquad \text{---------------------(A)}$$

Proof



Consider a circle of radius r, Let arc AB = angle x ,hence BC = angle (180 – x).
Now area ΔABC = (½)BC.AB = (1/2)AC.BD
Hence, 1/BD = AC/ AB.BC

Now x = arc AB > chord AB ,and, (180 – x) = arc BC > chord BC, and BD = r sin x
hence,1/(r sin x) > 2 r / x.(180 – x)
Let 1/(r sin x) = p[2 r / x.(180 – x)] +q for some real numbers p and q,

Therefore, $\text{rsin x} = \frac{x(180-x)}{[2pr + qx(180-x)]}$ ---------------------------(B)

Put $x = 30^0$,we get, ½ r = 30.150 / [2p r +30.150 q]

Put $x = 90^0$,we get, r = 90.90 / [2p r +90.90 q],

Solving these two equations, $p = 40500 / r^2$, and q = -1/4r, and putting in (B) and solving, we have finally;

$$\text{Sin x} = \frac{4x(180-x)}{(40500-x(180-x))}$$

The following table states the values of sine functions for some angles, in degrees calculated from formula (A):

Sine function	value from (A)	true value
Sin 10 ^{0}	0.17525	0.17535
Sin 20 ^{0}	0.34317	0.34202
Sin 30 ^{0}	0.5000	0. 5000
Sin 40 ^{0}	0.64183	0.64279
Sin 50 ^{0}	0.76471	0.76604
Sin 60 ^{0}	0.86486	0.86603
Sin 70 ^{0}	0.93903	0.93969
Sin 80 ^{0}	0.98461	0.98481
Sin 90 ^{0}	1.00000	1.00000

(ii) When x is in radians, arc AB = x and arc BC = π – x,

To prove, $$\text{Sinx} = \frac{16x(\pi-x)}{[5\pi^2-4x(\pi-x)]}$$

Proof

From eq. (B),

$$\text{rsin x} = \frac{x(180-x)}{[2pr + qx(180-x)]}$$

as in above proof, we get $p = 5\pi^2 / 32\ r^2$ and q = -1/4r.

putting in (B),we get

$$\text{Sinx} = \frac{16x(\pi-x)}{[5\pi^2-4x(\pi-x)]} \quad \text{------------(C)}$$

Following table states the values of sine functions for some angles, in radians calculated from formula (C):

Sine function	value from (C)	true value
Sin (π /4)	0.70588	0.70170
Sin (π/5)	0.58716	0.58778
Sin (π /6)	0.50000	0.50000
Sin (π /7)	0.43432	0.43437
Sin (π /8)	0.38326	0.38328
Sin (π /9)	0.34311	0.34346
sin (π /10)	0.31034	0.30903
Sin (π /20)	0.15800	0.15643
Sin (3π /20)	0.45434	0.45399

(iii) When *n* is a natural number, then

$$\text{Sin}\left(\frac{\pi}{n}\right) = \frac{16(n-1)}{[5n^2-4n+4]} \qquad \text{----------------------(D)}$$

Proof

The proof is obvious if we put x = π / n. in (C)
Put *n* = 2 and we have Sin(π / n) = 1

The following table states the values of sine functions for some *n* calculated from formula (D):

Sine function	value from (D)
Sin (π/3)	0.8648
Sin (π /4)	0.7058
Sin (π /5)	0.5871
Sin (π /6)	0.5000
Sin (π /7)	0.4343
Sin (π /8)	0.3832
Sin (π /9)	0.3431

(iv) put x = π + θ in eq.(C), we get, $\text{Cos}\,\theta = \frac{(\pi^2-4\theta^2)}{(\pi^2+4\theta^2)}$

to verify, put θ = π / 3, and we get cos π / 3 = 1 / 2.

(v) another result: Also putting x = 90 + y in (A), we have,

$$\text{Cos}\,\theta = \frac{4(90+y)(90-y)}{[40500-(90+y)(90-y)]}$$

Or, $$\text{Cos}\,\theta = \frac{4(8100-y^2)}{[32400+y^2]}$$

Another proof

This results that cos y = f(y^2), Now cos y function decreases as y increases for 0 to 90^0.
He suggests that cos y ∝ $1/y^2$.

As cos y vanishes for finite values of y**,** we subtract k, a positive constant and assume that

$$\cos\ y = \left[\frac{a}{y^2+b}\right] - k \text{ ....(1)}$$

where *a*, *b* are constants.

These three unknowns are obtained from (1) by using the following results:
at y = 0, cos y = 1, at y = 60^0, cos y = ½ and at y = 90^0,, cos y = 0,

This gives a = 162000, b = 32400 and k = 4

Hence, $$\cos\ y = \left[\frac{162000}{y^2+32400}\right] - 4$$

that is, $$\cos\ y = \left[\frac{32400 - 4y^2}{32400 + y^2}\right]$$

or, $$\cos\ y = \frac{4\,(8100-y^2)}{[32400+y^2]}$$

2. Bhaskara II, (1114-1185)

In his text, *Sidhant Shiromani*, there is a chapter, Jyotpatti (which means construction of sine functions). There he has stated another formula for finding the value of a sine function.
Consider a regular polygon of n sides ,with each side Sn, inscribed in a circle of diameter D. then Sn = D.sin (π / n) .

The table below is for sine functions, given by Bhaskara II, for some angles using above formula for *n*.

Sine function	value from (D)	true value
Sin (π/3)	0.86602	0.866025
Sin (π /4)	0.7071	0. 7071
Sin (π /5)	0.5877	0.58778
Sin (π /6)	0.5142	0.5
Sin (π /7)	0.4337	0.4338

Sin (π /8)	0.3826	0.38268
Sin (π /9)	0.3419	0.3420
Sin(π / 18)	0.17525	0.17535

In general, the maximum error is 0.0001.

Unfortunately, all these values of sine functions, given by either Bhaskara I or Bhaskara II do not satisfy the accuracy needed for astronomy.

In Jyotpatti, Bhaskara II states a Sanskrit verse translated as, "radius R of a circle, multiplied by 5878 and divided by 10000 gives R.jya 36°", where an arc of radius R subtends an angle of 36° at is centre.

$$\text{Jya } 36^0 = \sqrt{\frac{1}{8}\left(5R^2 - \sqrt{5R^4}\right)}$$

That is, sine 36° = 5878 / 10000 = 0.5878.

Also Bhaskara 2 states "jya18 = $\sqrt{\frac{1}{4}\left(\sqrt{5}-1\right)}$ = 0.17525 approximately

1. There are equivalent forms of sine rule (A) given by other mathematicians:

(i) In the text *Brāhmasphuṭa Sidhānta* of Brahma Gupta (6th C.), a similar result is given in stanzas, 22-23, Chapter 14, p.577-578
The fourteenth chapter has the couplets:

भुजकोट् यंशोन गुणा भार्धांशास्तच्चतुर्थ भागोनैः ।
पञ्चद्वीन्दु खचन्द्रैर्विभाजिता व्यासदल गुणिता ।। २३ ।।
तज्ज्ये परमफलज्या संगुणितास्तत्फले विना ज्याभिः ।
इष्टोच्चनीच वृत्त व्यासार्धं परमफल जीवा ।। २४ ।।

(*Brāhma Sphuṭa Siddhānta*, XIV, 23-24)

Meaning

Multiply the degrees of the bhuja and the koṭi by degrees of half a circle diminished by the same. (The product so obtained) be divided by 10125 lessened by the fourth part of that same product. The whole multiplied by semi-diameter givens the sine .....

i.e. $\text{Sin x} = \frac{4x(180-x)}{(40500-x(180-x))}$

Which is equivalent to Bhāskara I’s rule.

(ii) In the text *Lilavati* of Bhāskara II, a similar result is given in Sanskrit stanza, 205, page 142, Marathi language ed., 2014. The concerned stanza is:

चापोननिघ्न परिधिः प्रथमाह्वयः स्यात्
पञ्चाहतः परिधिवर्ग चतुर्थभागः ।
अद्योनितेन खलु तेन भजेच्चतुर्घ्नं-
व्यासाहतं प्रथममाप्तमिह ज्याका स्यात् ।।

(*Līlāvatī*, *Kṣetravyavahāra*, *śloka* No. 48)

Meaning

Circumference less (a given) arc multiplied by that arc is prathama. Multiply square of the circumference by five and take its fourth part. By the quantity so obtained, but lessened by prathama, divide the prathama multiplied by four times the diameter. The result will be chord (i.e. pūrṇa jyā or double-sine) of the (given) arc.

(iii) In the text *Ganit Kaumudi* of Narayan Pandit,(1356 ad.), a similar result is given in Sanskrit stanzas, 69-72, ch.4, p.37-38,
As in case of *Vaṭeśvara Siddhānta*, two forms of the rule occur here in the chapter called *Kṣetravyavahāra* as follows as follows:

वृत्त्यर्धं धनुरुनितं स्वगुणितं तेनोनयुक्ते क्रमाद्
वृत्त्यर्धं च वृतिश्चते स्वगुणते तौ गुण्यहाराह्वयौ ।
व्यासे गुण्यहते हराङ्घ्रि विहृते ज्यास्यादथाद्य ज्यया-
ऽऽसन्ना ज्या रहिता ग्रहाख्य गणितेस्युर्व्यास खण्डानि च ।।
अथवा
वृत्तेधनूरहित निघ्न वृतिर्द्विधा तां
व्यासाहतां च विभजेदितराङ्घ्रिहीनैः ।
वृत्त्यङ्घ्रि वर्ग गुणितै विषयैश्च जीवा
स्यात् खेचराख्य गणितेऽप्युपयोग एषः ।।

(*Gaṇita Kaumudī*, *Kṣetravyavahāra*, 69-70)

Meaning

Multiply to itself half the circumference less (a given) arc. The quantity so obtained when respectively subtracted from and added to the squares of half the circumference and circumference respectively gives the Numerator and Denominator. Multiply the diameter by the Numerator and divide by one-fourth of the Denominator to get the chord …

Multiply the circumference less the given arc by circumference and put the result in two places. In one place multiply it by the diameter and divide the result by five times the square of the quarter circumference less the quarter of the result in the other place. The final result is chord ...

(iv) In the text *Graha-Lāghav* of Ganesh Daivadnya,(1520 ad), a similar result is given in Sanskrit stanza 23, page 22, 1856 ed. (in Marathi language)

In this work the rule is given in various modified forms adopted for particular cases. The relevant text from *Ravicandrae-spaṣṭādhikāra* for one such case is:-

बिधोः केन्द्र दोर्भाग षष्ठोन निघ्नाः
खरामाः पृथक् तन्नखांशोनितैश्च ।
रसाक्षैर्हृतास्ते लवाद्यं फलं स्याद्रवीन्द्रू
स्फुटौ संस्कृतौ स्तश्च ताभ्याम् ।। ३ ।। (*Grahalāghava*, II, 3)

Subtract the sixth part of the degrees of the bhuja of the moon from 30 and multiply the result by the same sixth part. Put the product in two places. By 56 minutes the twentieth part of the product in one place divide the product of the other place. The result is the mandaphala.

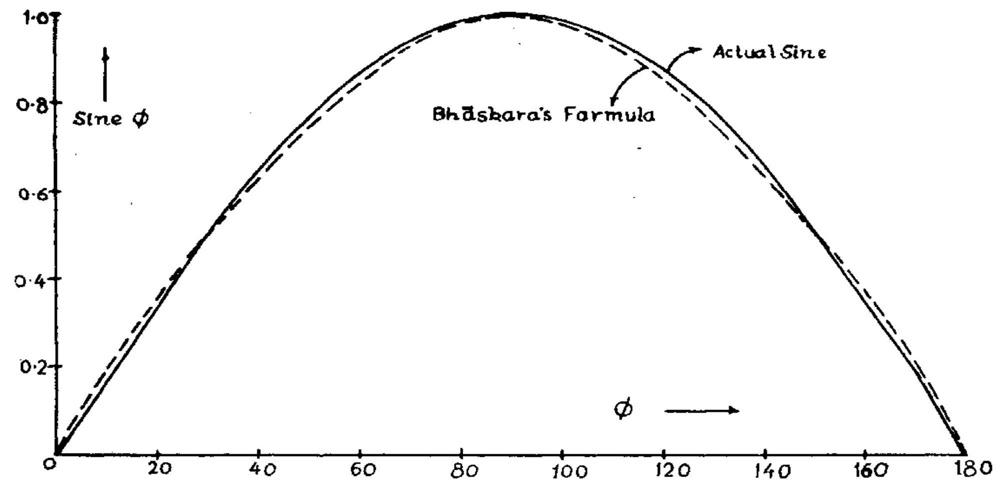
$$R\sin\varphi = \frac{\left(30-\frac{\varphi}{6}\right)\times\frac{\varphi}{6}}{56-\frac{1}{20}\left(30-\frac{\varphi}{6}\right)\times\frac{\varphi}{6}}$$

$$Sin\ \varphi = \frac{20\varphi(180-\varphi)}{R\{40320-\varphi(180-\varphi)\}}$$

$$= \frac{4\varphi(180-\varphi)}{40320-\varphi(180-\varphi)}$$

Since the value of the maximum *mandaphala* for the moon, i.e. the value of R for moon is 5 degrees approximately.

2. Two graphs, nearly overlapping, are plotted, to exhibit the approximate and actual values of a sine function given by both the Bhaskaras I and II.



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