HISTORY OF MULTIPLICATION

CLASSICAL PERIOD IN INDIAN MATHEMATICS.

Prabha S. Rastogi, Asst. Professor and Sandhya Nitin, Asst. Professor, JJT University, Rajasthan

Abstract

Indian mathematics seems to have developed independently of Chinese and probably even Babylonian mathematics. Mantras from the early Vedic period (before 1000 BCE) invoke powers of ten (from 100 to 1012) and provide evidence of the use of arithmetic operations such as addition, subtraction, multiplication, fractions, squares, cubes and roots. Seven distinct modes of multiplication were employed by the Indians, some of which are as old as 200 A.D. This paper discusses a few of the 'popular' methods of multiplication like Gomutrika, kapat-sandhi, khanda/bheda, sthana-khanda, tastha and others, in a bid to understand the way multiplication methods developed. We realise that different Indian mathematicians have interpreted the sulba-sutras differently, the major differences being the writing of the multiplier, multiplicand and intermediate products. Some mathematicians have also shown that product is unchanged whether multiplication starts from the units place or the highest place of the multiplier. Further, it was intriguing that the mathematics scholars of the olden times had devised methods so that multiplication tables from 2 to 9 were sufficient to solve any problem irrespective of the size of the multiplier.

Ancient Hindu mathematics has not grabbed the attention of scholars across the world even today; in fact, it is still considered to be shrouded in mystery. Absence of written evidence has made it vulnerable to misinterpretation. Ancient Indians used the "shruti" (oral) tradition in the "guru-shishya" practice, wherein knowledge was imparted directly by a teacher to his student. As time progressed, these sutras and verses, told orally by the guru (teacher) and memorised by the shishyas (students), were written down in cryptic verse in Sanskrit. The sutras are in a compact form (probably for ease of memorising), but without any detailed explanation, proofs or methodology. Lack of proof may be attributed to the fact that the system of education was direct verbal communication between teacher and student.

It seems that ancient Indian mathematics was extremely advanced, though the written material is only available from 900 BC onwards. The written sutras were in rhythmic poetic form or verses, as mathematicians were expected to be good poets so as to convey the subject to the students in a better yet compact way. Detailed commentaries of the sutras of Aryabhata, Brahmagupta, Bhaskara, among others have been given by Aryabhata II, Prthudaka Swami, Neelkanth Somaiya, Ganesa, etc. The commentaries are also written in Sanskrit verse without any detailed proofs. As Sanskrit verses have multiple, and often, hidden meanings, the interpretation and deciphering of the commentaries of the sutras is difficult.

Many of the translations of the commentaries were done by European scholars in the 1800s and could not catch the sentiments and essence of the original work and ideas of Indian scholars. Mathematics in India was developed primarily for religious practices which required knowledge of auspicious *muhurta* for their activities and various specific sizes and forms of *vedis* needed in *poojas*. Precise calculations and measurements were required for post-vedic religious practices which drove the development of higher level techniques of mathematical calculations in India. Hence most of the mathematicians were either priests or astronomers.

The teaching process was scientific yet concise. Young students were made to memorise the sutras (rules), and were taught how to apply these in real-life situations. Teachers guided and mentored students while they performed calculations on a *pati* that was covered with dust and used to write on with their forefinger or a stick. Re-use of the pati entailed simply smoothening the dust. The term *pati-ganita* may have come from this use of the pati (a slate) for doing *ganita* (arithmetic). Maybe the paucity of proper writing material or emphasis on direct a oral teaching methodology meant any treatise on Indian mathematics contained only brief statements of the formulae and results, which are barely comprehensible, without any explanation. An alternative reason could be that India was already facing invasions and repeated plundering from central Asian and Persian invaders. Ambiguously expressed sutras ensured that even if these fell in wrong hands, the knowledge would not be lost.

The main written sources of Indian mathematics are the Bhakshali manuscript (200 AD, author unknown), Surya Siddhanta (period and author unknown), Aryabhatiya (510 AD by Aryabhata), Brahmasphuta Siddhanta (628 AD by Brahmagupta), Trisatika (750 AD by Sridhara), Ganita-Sara-Sangraha (850 AD by Mahavira) Leelavati and Siddhanta Shiromani (1150 AD by Bhaskara II), Ganita Kaumudi (1356 AD by Narayana Pandit), Patisara (1658 AD by Ganesa) and many more. Indian mathematicians studied arithmetic with 20 different operations concerning mathematical calculations. One of the basic operations was multiplication, which has been mentioned by every mathematician from Brahmagupta to Bhaskara II, with each giving his own interpretation and modification. Aryabhata does not discuss any basic mathematical operations; Brahmagupta, who calls himself a bhakta (disciple of Aryabhata), was the first to mention multiplication explicitly.

Other mathematicians followed an approach similar to that of Brahmagupta to multiplication with minor modifications. Sridhara, however, has given another well-known method of multiplication called “Kapat-sandhi”. Ganesa’s commentary on Leelavati mentions a multiplication method also called kapat-sandhi that is similar to the currently popular Gelosia method. The multiplication methods were probably carried over to the West by the Persian mathematician Muhammad Al-Khwarizmi, who oversaw the translation of the major Greek and Indian mathematical and astronomy works (including those of Brahmagupta) into Arabic. Though the Sumerian, Egyptian and Greek civilisations used multiplication (for taxation, trade and land deals), their number systems did not include the symbol “zero”, nor did they use the place value concept. Al-Khwarizmi recognized the Indian numeral system as having the power and efficiency needed to revolutionize Islamic and Western mathematics. He called these ‘the Hindu numerals 1 – 9 and 0’, which were soon adopted by the complete Islamic world and later, the entire world.

Let us consider a few of these methods in some detail and see if any of them lead to the modern method. Aryabhata I does not mention the common methods of multiplication, probably because they were too elementary and too well-known to be included in a Siddhanta work. Brahmagupta has mentioned the following methods of multiplication: 1) *Gomutrika* method 2) *Khanda* or *Bheda* method and 3) *Ista-gunana* method.

1) The Gomutrika (Brahma Sphuta Sidhanta XII-55) method or the zig-zag method A

Ramswarup Sharma (1966) has given a brief description as of this sutra as follows:

The multiplicand repeated, as in gomutrika *as often as there are digits in the multiplier, is severally multiplied by them and (the results) added according to places; this gives the product. Or the multiplicand is repeated as many times as there are component parts in the multiplier.*

The following illustration is based on the commentary of Prthudaka Swami: $1224{\times}235$

The multiplication of $1224{\times}235$ starts with writing the numbers as shown below; the multiplier is written vertically downwards while the multiplicand repeated as many times as the multiplier digits and staggered to the right by one place. The first line of figures is then multiplied by 2, the process beginning at the units place, thus: $2{\times}4 = 8$; 4 is rubbed out on the ‘pati’ and 8 is written in its place and so on. After all the horizontal lines (1224) have been multiplied by the numbers on the left (vertical 235), the only numbers remaining on the pati are those which are written on the extreme right. These when added gives the product.

e.g. $1224{\times}235$	2	1 2 2 4	2 4 4 8
	3	1 2 2 4	3 6 7 2
	5	1 2 2 4	6 1 2 0
			Sum = 2 8 7 6 4 0

Another example (with carry-over)

2	1 9 9 8	3 9 9 6
7	1 9 9 8	1 3 9 8 6
		Sum = 5 3 9 4 6

2) *Khanda* or *Bheda* method – According to A. Ramswarup Sharma (1966),

This method is popular since Brahmagupta times. We can have two methods under this head.

The multiplier is broken into i) two or more additive parts or ii) two or more multiplicative factors. The multiplicand is then multiplied severally by these parts for ease of calculation and the results treated depending on whether additive or multiplicative parts were used. e.g.

i) When the multiplier is broken into parts which are additive, the results are added.

$13{\times}158 = (10 + 3){\times}158 = 1580 + 474 = 2054$

Or $3\times158 = (6+7)\times158 = (6\times158) + (7\times158) = 048 + 1106 = 2054$

ii) When the multiplier is broken into two or more multiplicative factors, repeated multiplication is carried out. e.g. 72×273

$$72\times273 = 4\times3\times6\times273 = 4\times3\times(6\times273) = 4\times3\times1638 = 4\times4914 = 19656$$

These methods of multiplication by parts are found among the Arabs and Italians (who called them 'Scapezzo' and 'Repiego' respectively), having obtained them from people of India.

3) Ista-Gunana method or the Algebraic method

A brief description of ista-gunana as given by A. Ramswarup Sharma (1966).

The multiplicand is multiplied by the sum or the difference of the multiplied and an assumed quantity and from the result the product of the assumed quantity and the multiplicand is subtracted or added. (Brahma Sphuta Sidhanta XII-56)

This method involves either i) adding or ii) subtracting an assumed number. The assumed number is chosen to give two numbers with which multiplication is easier than with the original multiplier. The two ways are stated below –

i) $93\times17 = (93+7)\times17 - 7\times17 = 1700 - 119 = 1581$
ii) $93\times17 = (90+3)\times17 = 90\times17 + 3\times17 = 1530 + 51 = 1581$

This method was used extensively by Arabs and Europeans, probably taken out from this country by Khwarizmi.

Another method, called the *Sthana-Khanda* method, has been popular since the Brahmagupta period. According to A. Ramswarup Sharma (1966), the present day multiplication method resembles the *sthana-khanda* and the *Gomutrika* methods most closely. Let us look at this stana-khanda method in some detail now.

The *Sthana-Khanda* method: Let us understand the meaning of this name: "sthana" means place and "khanda" means part; so this method is based on the separation of the digits of the multiplicand or of the multiplier. According to Datta and Singh (1935), Bhaskara II describes this method as follows

*Multiply separately by the places of figures and add together according to places (*yatha sthanam sahit*) to get the result.* e.g. 235×14. Following are the three different ways in which the intermediate steps of this multiplication are written:-

i)
$$\begin{array}{cccc} & 2 & 3 & 5 \\ 1 & 4 & & \\ \hline 2 & 8 & & \\ & 4 & 2 & \\ & & 7 & 0 \\ \hline 3 & 2 & 9 & 0 \end{array}$$

ii)
$$\begin{array}{c|c|c} 14 & 14 & 14 \\ 2 & 3 & 5 \\ \hline \end{array}$$
$$\begin{array}{cccc} 2 & 8 & 7 & 0 \\ & 4 & 2 & \\ \hline 3 & 2 & 9 & 0 \end{array}$$

iii)
$$\begin{array}{cccc} 2\,3\,5 & & 2\,3\,5 \\ 1 & & 4 \\ \hline & 9\,4\,0 & \\ 2\,3\,5 & & \\ \hline 3\,2\,9\,0 & & \end{array}$$

All the above methods were prevalent since the time of Brahmagupta. These methods use the concepts of “place value” and “carry over” extensively. We found explicit reference to the place value *(yatha sthanam sahit)* concept, however the carry over concept seems to be implied. Over a hundred years later (~710 AD), Sridhara suggested another method called the “Kapat-sandhi”, which is another well-known method of multiplication; explained by V. Kutumbshastri in his translation of Trishatika (2004). Mahavira in his treatise “Ganita Sara Sangraha” (Jain, 1963) gives a slightly different method and calls that kapat-sandhi. Aryabhata II, Bhaskara II, Sripati and Ganesa have also mentioned the kapat-sandhi method in their individual commentaries, though Ganesa’s approach is radically different. We discuss the kapat-sandhi method now. (“kapata” means "door" and “sandhi” means "junction"; hence “kapat-sandhi” means "the junction of doors")

According to Trishatika (Kutumbshastri, 1963), Sridhara describes kapat-sandhi method as follows:

Placing the multiplicand below the multiplier as in kapat-sandhi, multiply successively, in the direct or inverse order, moving the multiplier each time. This method is called kapat-sandhi. Aryabhata II, Sripati, Mahavira, Bhaskara II and Narayana use nearly the same words when describing their method of multiplication.

The kapat-sandhi method (Datta, 1935) discusses (i) the relative positions of the multiplicand and the multiplier and (ii) the rubbing out of figures of the multiplicand and writing the figures of the product in their places. Aryabhata II has even stated specific positions - *Place the first figure of the multiplier over the last figure of the multiplicand*; though he has not stated the name of the method. The position of the multiplier specifies the digit to be multiplied. Multiplication begins with the first digit of the multiplicand. e.g. $145\times13=?$

$$\begin{array}{ccc} & 1 & 3\\ 1 & 4 & \cancel{5}\end{array} \longrightarrow \begin{array}{cccc} & 1 & 3 & \\ 1 & 4 & 6 & 5\end{array} \longrightarrow \begin{array}{cccc} & 1 & 3 & \\ 1 & \cancel{4} & \cancel{6} & 5\\ \hline 1 & 5 & 8 & 5\end{array} \longrightarrow \begin{array}{cccc} 1 & 3 & & \\ 1 & \cancel{5} & 8 & 5\\ \hline 1 & 8 & 8 & 5\end{array}$$

Since 1, the higher digit of multiplier, is on 5, 5 is the first digit to be multiplied. Multiply 5 by 3 to get 15, write 5 below 3 and carry over 1; then 5 is multiplied by 1 to get 5. Add the (carried over) 1 to 5 and we get 6. The mathematicians of the olden days used the “pati” covered with dust, so rubbing and rewriting any number was easy. After the first multiplication the numbers on the pati read 1465. Next shift multiplier so 1 is on 4 so 4 must be multiplied by 13. $4\times3=12$, where 2 is written below 6; $6+2=8$ and 1 gets added to 4 to give 5 to give 1585 after multiplication of 4 is over. The numbers now read 13 above 1585, with 1 over 1 as the last digit of the multiplicand has to be multiplied. $1\times3=3$, which gets added to 5 to give 8 while $1\times1=1$, which gives the final product as 1885. What is fascinating is that such a method of multiplication means it was necessary to memorise multiplication tables from 2 to 9 only.

Aryabhata, Brahmagupta and Shridhara have all commented that the kapat-sandhi method can be done in the direct (as shown above) or in the inverse order; in fact it is less prone to mistakes in the inverse method. Here multiplication begins with the last digit of the multiplicand. We take the same example. $145\times13=?$

$$\begin{array}{c}13\\ \cancel{1}45\end{array} \longrightarrow \begin{array}{c}13\\ 1345\end{array} \longrightarrow \begin{array}{c}13\\ 1\cancel{3}\cancel{4}5\\ \hline 1825\end{array} \longrightarrow \begin{array}{c}13\\ 1825\end{array} \longrightarrow \begin{array}{c}13\\ 18\cancel{2}\cancel{5}\\ \hline 1885\end{array}$$

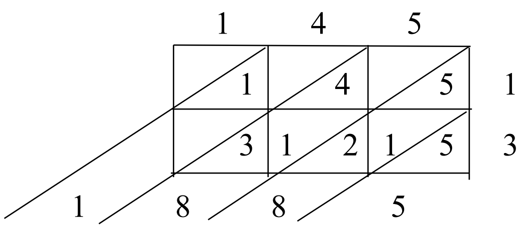
$3\times1=3$, 1 is wiped and 3 is written in its place; and $1\times1=1$, and 2nd set of number remain on the pati. Shifting 13 to the right by one place gives us the 3rd display on the pati.
Now, $3\times4=12$, so 4 is wiped and 2 written, 1 carried over. Now add the 3 to $1\times4=4$, and 1 (carried over) giving 8 instead of 3; shift 13 to the right to give the 4th set of numbers. Last digit of the multiplicand has to be multiplied by 13. We get $3\times5=15$, wipe 5 from units place to write final product 5, carry 1; and in place of 2 we get 8 as $1\times5=5$, which gives $5+2+1$ (carried over) = 8. Thus,final answer is 1885. Actuallythe carried over numbers were written in a separate portion of the pati and rubbed after product is found.

This inverse method of kapat-sandhi was more popular than the direct one. We emphasize that the position and movement of the multiplier play animportant part in this method. This kapat-sandhi method of multiplication also travelled with the Arabs (who learnt the decimal arithmetic from Indians), and thence to Europe. It appears in the works of Al-Khwarizmi, Al-Nasavi, etc; Al-Nasavi called it “the method of the Hindus” (Datta and Singh, 1935). In Europe, the method is found reproduced in the work of Maximus Planudes. (MAA Publication, 2012. Datta and Singh 1935).

D.E. Smith in his History of Mathematics (1958) states that another method of multiplication called the *Gelosia Method* appears in the “Ganita-manjari” (16th century) as the kapata-sandhi method. Ganesa in his commentary on the Lilavati also calls the gelosia method as the kapat-sandhi method. Whether the gelosia method went from India to the Arabs or from Arabs to Indians is not clear as of now; or is gelosia method just a smarter way of writing the kapat-sandhi is not clear.

The only available description of the method is: Draw as many compartments as there are places in the multiplicand and below these as many as there are places in the multiplier; the oblique lines produced in all compartments denote tens and units place. Multiply each place *of the multiplicand, by the places of the multiplier which are one below the other* and set the results in the compartments. The sum taken obliquely on both sides of the oblique lines in the compartments gives the product. This is the “kapata-sandhi”.

Let us solve the same earlier problem: 145×13 (The drawing of boxes as gelosia method)



This method has been mentioned by Sridhara, Mahavira, Sripati and some later writers as the "tastha" method in which the multiplier is stationary (tastha*)* (Ganesa, pg. 3). The tastha method is algebraic and has been compared to "tiryak-gunana" or "vajrabhyasa*"* (cross multiplication) used in algebra (Colebrooke, pg. 171).

The tastha method - Ganesa (c. 1545) has explained this in some detail as follows: *That method of multiplication in which the numbers stand in the same place, is called "tastha-gunana". After setting the multiplier under the multiplicand, multiply unit by unit and note the result underneath. Then as in vajrabhyasa, multiply unit by ten and ten by unit, add together and set down the result in the line. Next multiply unit by hundred, hundred by unit and ten by ten, add together and set down the result as before; and so on with the rest of the digits. This being done, the line of results is the product.*

This method was known to the Indian scholars of the 8th century, or earlier. The method seems to have travelled to Arabia and thence to Europe and is stated to be "more fantastic and ingenious than the others but cannot be learnt without the traditional oral instructions".

On re-reading the statement on the tastha method by Ganesa, it is not difficult to realise that the intermediate addition of the various numerals (*multiply unit by ten and ten by unit, add together, or multiply unit by hundred, hundred by unit and ten by ten, add together)* gives in fact, the exact methodology of the vertically crosswise method!

References

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