The Side of a Regular Polygon – Lilavati's Approach

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INTRODUCTION

The text *Lilavati* is one of the finest contributions of a great mathematician Bhaskara (also known as Bhaskaracharya of 12th Century) and one of the most popular books on ancient mathematics. Lilavati was the name of his daughter and Bhaskara taught her topics of mathematics relevant to the12th century. The year 2014 was the 900th birth year of Bhaskara and was celebrated thought India. Lilavati is the only text in ancient Indian mathematics which contains arithmetic, algebra, geometry, combinatorics, trigonometry and astronomy using interesting puzzles and games. The original script is in Sanskrit.

In this paper, we present the Lilavati approach of finding the side of a regular polygon of *n* sides (*n* from 3 to 9) *inscribed* in a circle with known diameter. As a matter of fact, any triangle can be circumscribed, but not all polygons can be circumscribed. Bhaskara knew this and hence he dealt only with circumscribable polygons. It is to be appreciated how Bhaskara obtained the results with no tools available in 12th century.

Lilavati contains 261 stanzas (*shlokas*). Of those, stanzas 144 to 223 are devoted to geometry. After 223, trigonometry and other topics are discussed.

From shloka 195, start the problems on figures circumscribed in a circle with known diameter. This part contains length of a chord, angle made by the arc at the center, etc. Our focus in on shlokas 202, 203 and 204.

Consider the following three *shloka*:

त्रिद्व्यंकाग्निनभश्चन्द्रैस्त्रिबाणाष्टयुगाष्टभिः ।
वेदाग्निबाणखाश्वैश्च खखाभ्राभ्ररसैः क्रमात् । ।202 । ।

बाणेषुनखबाणैश्च द्विद्विनंदेषुसागरैः ।
कुरामदशवेदैश्च वृत्ते व्याससमाहते । ।203 । ।

खखखाभ्रार्कसंभक्ते लभ्यन्ते क्रमशो भुजाः ।
वृत्तान्तत्र्यस्त्रपूर्वाणाम् नवास्त्रान्तं पृथक् पृथक् । ।204 । ।

The collective meaning of these stanzas is as follows:

"Multiply the diameter of a given circle by the coefficients of 103923, 84853, 70534, 60000, 52055, 45922, 410031 in order.

Then divide each of these products by 120,000. The results are the sides of regular polygons of sides 3, 4, 5, 6, 7, 8, and 9 respectively"

It is surprising how these coefficients were obtained by Bhaskar. This suggests the generalization of the above result as

$$S_n = \frac{(D \times P_n)}{120{,}000}$$

where, S_n = the side of a regular polygon of n sides ,

P_n = the coefficients mentioned above paragraph taken in order

D = diameter of the circle.

In particular, when D = 120,000, $S_n = P_n$.

Using above formula given in Lilavati , following are the sides of a regular polygon of n sides. Here, D stands for the diameter of circumscribing circle.

for n = 3, S_3 = (D x103923)/ 120000 = D x 0 .8660,

similarly, the other values are:

for n = 4, S_4= D x 0.7071,

for n = 5, S_5 = D x 0.5878,

for n = 6, S_6 = D x 0.5000,

for n = 7, S_7 = D x 0.4339,

for n = 8, S_8 = D x 03420,

for n = 9, S_9 = D x 0.3826,

The method of deriving these coefficients P_n is not given in Lilavati. One of the ancient mathematicians, Ganesh Daivadnya, (16th century), stated a result for finding the coefficients as :

$P_n = 120{,}000 \times \sin\left(\frac{180}{n}\right)$, provided we can find the value of sine ratio of $180/n$.

For n = 3, 180/3 = 60

and sin 60° = $\sqrt{3}/2$ = 0.8660 = (8660 $\times$ 12)/120000

= 103923/120000

Hence P_3 = 12000 $\times$ (103923/120000) = 103923

and then S_3 = D $\times$ 0.8660 which same as above.

For n = 5, 180/5 = 36 and sin 36° = 0.5878

$= (5878 \times 12)/120000$

$= 70536/120000$

$P_5 = 12000 \times (5878/120000) = 5878$

so that $S_5 = D \times 0.5878$ which same as above.

For $n = 7$, $180/7 = 27.71$ and $\sin 0.2571^{o} = 0.4339$,

so that $S_7 = D \times 0.433$ which same as above.

The other values for n be similarly obtained. The only difference in Ganesh Daivadnya's formula is that the value obtained differs after the 6th decimal place.

It seems that the diameter of a circle, Bhaskara has imagined, is 120,000. Ganesha used trigonometry to find these coefficients

For finding the coefficients P_n for $n = 3, 4, 6, 8$ and 9, the exact value of sine function can be found out. The difficulty arises for $n = 5, 7$, and other values of n as the sine ratios are irrational and the exact value P_n, and thereby S_n cannot be found. In the above cases we have taken the approximate values.

CONCLUSION

It may be possible to find other values of P_n by using various methods of numerical analysis.

Many editions and commentaries have been published on the text Lilavati. Each author has given their view about finding the side of a regular polygon, but very few have stated the origin of the coefficient P_n.

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