Bharathi Krishna Tirtha's Vedic Maths and Early Indian mathematics: Comparison of the Fundamental Arithmetic Operations

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ABSTRACT

Indian mathematicians of the past millennia describe twenty operations of arithmetic (additionally 8 determinations). Of these, Addition, Subtraction, Multiplication, Division, Squaring, etc. are considered fundamental operations. In a recently published work (Prasad, 2015)[1], Multiplication techniques from early Indian mathematics were compared against the Multiplication techniques contained in Bharathi Krishna Tirtha's methods in Vedic Mathematics (henceforth VM). Several similarities were found, including one-to-one match with certain techniques, notably, the 'Vertically and Crosswise' method in VM called *Urdhva Tiryagbhyam*, and named *Tastha* by the early Indian mathematicians. The current paper is an extension of this comparative analysis to additional fundamental arithmetic operations viz. Addition, Subtraction and Division. Furthermore, the operation of Squaring is also included in the current analysis. As in the previous work, these arithmetic operations are compared and contrasted within the two systems, viz. early Indian mathematics and VM. The comparison reveals some similarities in the two systems, such as Left-to-Right and Right-to-Left processes for Squaring, Addition and Subtraction operations. Furthermore, the idea of the operation of Division being an inverse of Multiplication is seen in both systems. Following these comparisons, some conclusions are drawn.

Keywords: Indian mathematics, Vedic Mathematics, History, Arithmetic operations

INTRODUCTION

There are 8 fundamental arithmetic operations defined in early Indian mathematical treatises (Datta and Singh, 2004). These are: Addition, Subtraction, Multiplication, Division, Squaring, Cubing, Square-root and Cube-root. In the previous paper (Prasad, 2015) the operation of Multiplication was studied. The different ways of Multiplication were compared and contrasted between early Indian and VM systems. Several similarities and few differences were found in the Multiplication methods in the two systems. In this paper we compare four more fundamental operations – Addition, Subtraction, Division and Squaring.

Although mathematical thought in Indic civilization goes to Vedic times (Datta and Singh, 2004), the current paper deals only with early Indian mathematics. Here, early mathematics is defined as starting from Aryabhatta I (~500 C.E.). In contrast, VM has an intriguing history and brings forth an interesting case for assigning the chronology. It is said to be reconstructed in the early 20th century by Bharathi Krishna Tirtha, following a deep study of the Vedic texts, which in turn dates back to several millennia[2] before the Common Era.

This comparative work, like previously (Prasad, 2015), is mainly taken from (Datta and Singh, 2004) for the early Indian mathematics, and several sources for the VM methods [Raval, 2014;

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[1] Note: The word 'Ancient' was changed to 'Early' in the presentation.

[2] For instance, astronomical references in the *Rig-veda* puts it to before 6000 B.C.E.

Nicholas, Pickles and Williams, 1982; Vedic Mathematics Academy, 2014-15], chief among them being Tirtha's introductory book on VM (Tirtha, 2001). Given the space limitations for this paper, numerical examples only for select techniques are provided. The interested readers are encouraged to consult the bibliography for solved examples.

1. SQUARING

Due to its proximity to the Multiplication technique, the methods of Squaring are described first.

1.1 Early Indian methods

Five different Squaring methods have been presented by early Indian mathematicians. These methods are for whole numbers, but easily extended to real numbers – the place value system to be meticulously followed for all cases. It must be emphasized that in general the place value system is critical to both systems. Furthermore, any of these methods may be used for squaring the fractions (squaring fractions is simply squaring the numerator and the denominator separately {Brahmagupta (Datta and Singh, 2004)}). Note that these are separate from multiplying the number with itself which is a valid Squaring method. These methods are collected in Table 1, along with their corresponding authors.

Table 1. Squaring methods in early Indian mathematics (Datta and Singh, 2004).

S.N.	Squaring Method	Author
1	General method	Aryabhatta, Brahmagupta
2	$n^2 = (n-a)(n+a) + a^2$	Brahmagupta, Mahavira, Bhaskara II, Narayana
3	$(a+b)^2 = a^2 + b^2 + 2ab$	Bhaskara II, Mahavira
4	$n^2 = 1 + 3 + 5 + \cdots n\ terms$	Sridhara, Mahavira
5	$A^2 = (a-b)^2 + 4ab$	Narayana

Here, methods #3 and #5, and in some ways method #4, require breaking the number to be squared into smaller parts. Method #2 can be thought of as a variant of methods #3-#5, where a number is also chosen, but not as a part of the original number. These methods are self-explanatory and are not discussed further. The general method #1 has been described by several mathematicians and the process discussed by Mahavira is reproduced from (Datta and Singh, 2004).

"Having squared the last (digit), multiply the rest of the digits by twice the last, (which is) moved forward (by one place). Then moving the remaining digits continue the same operation."

A worked example is shown below.

132^2 from right to left:

$$\begin{array}{ccc} - & - & 4 \\ 1 & 3 & 2 \\ - & 4 & - \end{array} \xrightarrow{13x4=52} \begin{array}{ccc} 5 & 2 & 4 \\ 1 & 3 & 2 \\ - & - & - \end{array} \xrightarrow{2\,removed} \begin{array}{ccc} 5 & 2 & 4 \\ 1 & 3 & - \\ - & - & - \end{array} \xrightarrow{3^2=9\,and\,3x2=6\,added}$$

$$\begin{array}{cccc} 1 & 4 & 2 & 4 \\ 1 & - & - & - \\ 6 & - & - & - \end{array} \xrightarrow{1^2=1\,and\,6x1\,added} 1\quad 7\quad 4\quad 2\quad 4$$

Notice the square of 2^2 and 2x2=4 (placed below 3) in the first step.

132^2 from left to right:

$$\begin{array}{ccc} 1 & - & - \\ 1 & 3 & 2 \\ - & 2 & - \end{array} \xrightarrow{32x2=64,1\,removed} \begin{array}{ccc} 1 & 6 & 4 \\ - & 3 & 2 \\ - & - & - \end{array} \xrightarrow{3^2=9\,and\,3x2=6\,added,3\,removed} \begin{array}{ccc} 1 & 7 & 3 \\ - & - & 2 \\ - & - & 6 \end{array} \xrightarrow{6x2=12\,added\,6\,removed}$$

$$\begin{array}{cccc} 1 & 7 & 4 & 2 \\ - & - & - & 2 \\ - & - & - & - \end{array} \xrightarrow{2^2=4\,added} 1\ 7\ 4\ 2\ 4$$

Notice the square of 1^2 and 1x2 = 2 (placed below 3) in the first step.

The movement of digits above are based on place-value rules.

1.2 VM methods

Duplex method: This is a general method applicable to all squaring sums. The Duplex rule is that the digits, when by themselves, are squared, and when in a pair they are multiplied together and doubled. A digit equidistant from either side (left and right) is deemed ‘by themselves’. Using this rule, proceeding from left or right or right to left, any number may be squared. An example follows:

$239^2 = 4_1 2_4 5_5 4_8 1 = 57121$

Here, the subscripts are carried over and placed as required by the place-value rules. It can be shown that the elements of ‘Duplex’ is seen in ‘Vertically and Crosswise’ multiplication, and therefore, not surprisingly, the Duplex method of squaring resembles the general method of squaring given in Table I. Although the above example is that of an integer, note that this method is applicable to decimals as well.

Special methods: There are several different special methods of squaring in VM. This follows the overall scheme of this system, that is, there is a general method and multiple special methods for a given operation. Special methods for squaring is enlisted below.

When we have two numbers whose 1s place digit add up to 10 and the remaining digits (in higher place value) are the same, then ‘*Ekadhikena Purvena*’ sutra can be used.

Square of a number which is near a base (10, 100, etc.) is simply performed by '*Yavadunam Tavadunam Krtyacha vargayet*' (for number less than the base). The verse is easily adjusted for a number greater than the base by using the word '*Adhika*'. The technique is also extendable to multiples of the base (20, 300, etc.). Bachubhai (Raval, 2014)) correctly describes it as same as method #2 in Table I, calling it Sripati's method. Irrespective of associating the method with the exact name(s) of the early Indian mathematician(s), the critical point to note is the similarity of this method in the two systems.

2 ADDITION

2.1 Early Indian methods

The process of addition has been assigned several names in Indian mathematics – *Abhyasa*, *Misrana*, *Samkalana*, *Yoga*, etc. With such opulence of names for this process, it seems that 'Addition' would have been very well understood, and, as Datta/Singh describe, 'taken for granted'. Nevertheless, Aryabhatta II gives a formal definition – '*the making into one of several numbers*', while Bhaskara II gives the technique in his commentary on *Lilavati*: *Add the figures in the same places in the direct or the inverse order.* Note that the idea of place-value system is stressed again – 'figures in the same places' – and the direct and inverse order refer to starting from right (direct) or the left (inverse). Adjustments to be made if the count of the digits in a given column (place-value - 1s, 10s, etc.) exceeds 10.

2.2 VM methods

VM methods rely principally on the idea of completing the whole or identifying the deficit, the corresponding sutras being '*Yavadunam*' and '*Purnapurnabhyam*', respectively. Within the '*Yavadunam*', the idea is to look for digits that add up to 10. In the *Purnapurnabhyam* the idea is to once again complete the 10 for a given base (10, 20, …, 60, etc.) by looking for the deficiency for 10. There are multiple methods given in (Nicholas, Pickles and Williams, 1982) capturing different scenarios, and they all follow the two basic principles mentioned above. They also include the use of bar-numbers for addition.

While details of the addition process in early Indian mathematics is scanty (Datta and Singh, 2001), likely because it was considered trivial, it does not preclude the fact that the early mathematicians might have been using principles any different than what is described within the VM curricula.

3 SUBTRACTION

3.1 Early Indian methods

Much like the case for ‘Addition’, the process of ‘Subtraction’ too does not occupy any significant description in early Indian mathematical works. Moreover, the process of ‘Subtraction’ too was defined variously – *vyutkalita*, *shodhana*, etc. Once again, Aryabhatta II gives the definition of the process: *The taking out of some number from the sarvadhana (total) is subtraction, what remains is called shesa (remainder)*. The actual method is described by Bhaskara II: *Subtract the numbers according to their places in the direct or the inverse order*. The definition is practically the same as that for addition and requires no further explanation.

When all the digits in the minuend are greater than the corresponding digits in the subtrahend, the process is straightforward. However, when some or all the digits in the subtrahend are greater than the digits in the corresponding places in the minuend (except the digit in the highest place-value) then the process requires some explanation.

1000 – 360 (the same as in (Datta and Singh, 2001)). The method explained by Suryadasa is very similar to the ‘All from 9 Last from 10’ sutra used in VM. The adapted text from (Datta and Singh, 2001) explaining this method is reproduced below. The method is described for the direct process where the process starts from the units place.

…the figure of the subtrahend that cannot be subtracted from ten, the remainder is taken and this ten is deducted from the next place. In this way this ten is taken to the last place until it is exhausted with the last figure.

This technique is akin to the modern practise, which is restricted to the direct process. However, the early Indian method allows for the indirect process also. Notably, while both direct and indirect processes are used in both addition and subtraction, it is said that the inverse process is better suited for subtraction and the direct process for addition (Datta and Singh, 2004).

3.2 VM methods

These methods also allow for direct and indirect processes. The techniques, described next, include Tirthaji’s ‘Nikhilam’ and the use of vinculum (also called ‘Bar-numbers’), and are unique to VM.

3.2a Nikhilam

This method essentially simplifies the cumbersome and error-prone ‘carry-over’ method of subtraction used in contemporary mathematics. Here, ‘last’ refers to the non-zero digit at the unit’s place of the subtrahend.

$$\begin{array}{r} 5000 \\ -387 \\ \hline 4613 \end{array}$$

In the above, 5 is reduced by 1, yielding 4. The following 0s, from left to right, are replaced by 9, 9 and 10 respectively and subsequently, subtraction carried out.

Bachubhai (Raval, 2014) describes this process using the concept of 'Nikhileshwar' and 'Nikhil'. The Nikhileshwar/Nikhil concept is easily extended to perform subtraction between two numbers and is also useful when there is mixed operation of addition and subtraction.

3.2b Bar-number method:

The extension of 'Nikhilam' is used in the idea of bar-numbers. These can be defined as writing a number as a combination of addition and subtraction. An example of bar-number and its use in subtraction follows:

$$29 = 30 - 1 = 3\bar{1}$$

$$\begin{array}{r} 5000 \\ -387 \\ \hline 5\bar{3}\bar{8}\bar{7} = 4613 \end{array}$$

Note that converting from bar-numbers to normal number requires the 'Nikhilam' method.

4 DIVISION

Much like Addition and Subtraction, the operation of Division too was considered trivial and not many details are provided. Importantly though, the early Indian mathematicians looked at Division as an inverse of Multiplication, which Datta/Singh indicate to be the reason Division was considered trivial. Regardless, VM also considers Division to be the inverse of Multiplication.

4.1 Early Indian methods

Aryabhatta II gives some details on the technique, "*Perform division having placed the divisor below the dividend; subtract from (the last digits of the dividend) the proper multiple of the divisor; this (the multiple) is the partial quotient, then moving the divisor divide what remains, and so on.*" An example with this technique is provided below. The sum is 6306/29.

$$\frac{6306.000}{29} \rightarrow \frac{506.000}{29} \rightarrow \frac{216.000}{29} \rightarrow \frac{130.000}{29} \rightarrow \frac{140.000}{29} \rightarrow \frac{240.000}{29} = 217.448$$

Some description of the process is provided. Here the divisor is aligned with the position of the digit, placed at the highest value. That is, the divisor moves to the right after every step. The final answer (217.448) is given in the end. In the first step 29x2 (divisor and the first digit of the quotient) yields 58, which when subtracted from 63 returns 5. This remainder along with the remaining digits in the number is then written as $\frac{506.000}{29}$. Digit 2 (divisor's highest placed digit) is placed directly under 5. The next digit in the quotient is 1, which when multiplied by the divisor yields 29. Subtracting 29 from 50 returns 21, and after following the same process as before, we get

$\frac{216.000}{29}$. These steps are repeated as long as required. Note that the method comes under the 'Inverse Process' and, in principle, contemporary mathematics uses the same method.

4.2 VM methods

VM offers several options for division. Bachubhai (Raval, 2014) provides three different techniques, all very similar to each other – 'On the Flag' (Dhwajanka), 'Transpose' *(Paravartya)*, and 'Nikhilam' methods. There is an additional 'Argument' method described by (Tirtha, 2001) and also followed by VM courses/lectures. Moreover, halving, division by 4 as twice repeated halving process, etc. are offered in standard VM courses, which are trivial from a mathematics viewpoint, and need not concern us here. Limited space does not allow description of all the methods mentioned and we therefore focus only on one of these techniques. The popular and generic Dhwajanka method is illustrated with an example (again, 6306/29).

4.2a Dhwajanka

$2\ ^{9}|6_{2}3_{3}0_{7}6._{5}\ 0_{6}0_{8}0 = 217.448$

Here, 2 is the active divisor and 9, written as a superscript, is 'placed on the flag'. The subscripts are the remainders after each division operation by the active divisor. For example, in the first step, 2x2 (active divisor and the first digit of the quotient) returns 4. This, when subtracted from 6, leaves a remainder of 2. The flag digit 9 is multiplied by the quotient 2 to give 18, which when subtracted by 23 (remainder) yields 5 as the new remainder. Now, 5 divided by the active divisor 2 gives the next quotient digit 1, which following subtraction leaves 3 as the remainder. This is appended to the next digit of the dividend, returning 30. The process is repeated as long as necessary.

Note that vinculum could also have been used, in which case the divisor would change to $3\bar{1}(= 29)$, which further simplifies the sum. With $\bar{1}$ on the flag, note the smaller remainders (subscripts).

$$3\ ^{\bar{1}}|6_{0}3_{2}06._{1}\ 0_{2}0 = 217.448$$

It should be noted – well known to the VM practitioners – that the number of digits on the flag is a matter of choice. For instance, a divisor with 4 digits can have 1, 2, or 3 digits on the flag.

4.2b Nikhilam method

This method is used when the divisor is close to a base, but smaller than the base. Hence it can be classified as a 'specific' method of division. Such 'specific' and 'general' method of arithmetic operation has been reported previously (Prasad, 2015).

The Nikhilam method may seem similar to the vinculum method. However, the essential difference is that while the vinculum method aims to reduce large numbers to smaller numbers (such as 9 in 29), the 'Nikhilam' method uses the remainder from when the divisor is subtracted from the base. For instance, the divisor 9 returns an active divisor of 1 after subtracting the divisor from the base 10. This method allows the operation of addition to be used during division and makes division easier for divisors close to base (100, 1000, etc.). Indeed, this can be a potential drawback of this process for smaller divisors far from the base, which makes the process cumbersome. This leads us to the '*Paravartya Yojayeta*' (Transpose and Apply) method.

4.2c Transpose method

Here the divisor is close to a base, but exceeds it. This method also uses the remainder in a divisor after subtracting the divisor from the base. For instance, for the divisor 12, active divisor becomes $\bar{2}$ after subtracting from the base, 10. This method, with bar-numbers engaged[3], also allows addition to be used during division.

Finally, the 'Argument' division method is essentially using the *'Urdhva Tiryak'* sutra in reverse. This technique is most suitable for algebraic divisions.

CONCLUDING REMARKS

Four fundamental arithmetic operations of Addition, Subtraction, Division and Squaring were compared and contrasted vis-à-vis the early Indian and VM systems. Not much details is found in early Indian works about the process of Addition, Subtraction and Division – thought to be trivial operations, and therefore not requiring much commentaries. As was found with the multiplication methods, there are significant similarities in the operations employed between the two systems. There are some techniques in VM method which seem to be unique to it. Though not elaborated in this article, these methods are also applicable to algebra and fractions.

Squaring: There are multiple ways to square a number in both systems and similar techniques were found. Furthermore, the 'Duplex' method used in VM is very close to the general method of squaring in the early Indian system. Both systems allow direct and indirect process.

Addition: While several different techniques are presented in VM system, these methods are but a variation of one fundamental principle – completing a '10'. The processes vary in completing this 10, either as a sum of digits totalling '10' or looking for a deficiency to complete a '10'. In both systems, the operation can be performed by direct or indirect process.

Subtraction: The 'Nikhilam' technique seems common to both the systems. Although by stating this rule explicitly, VM system makes it simpler. Use of bar-numbers to subtract seems unique to the VM method.

Division: 'On the Flag' sub-sutra seems unique to VM. The similarity between VM and early Indian method is that both the systems view Division as reverse of Multiplication and Division techniques were developed based on this notion. In other words, 'Division being reverse of Multiplication' was not merely a theory. Additionally, Tirtha's book discusses the Nikhilam and the Paravartya methods (Tirtha, 2001) quite extensively, providing the scenarios under which one is more suitable than the other.

Critically, both the systems require building on a strong foundation of initial operations before the next operation is tackled. For instance, all 'Addition' processes must be grasped and mastered which help make subsequent 'Multiplication' methods easier. Conversely, weaknesses in 'Addition' methods would make 'Multiplication' and 'Division' processes cumbersome.

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[3] Note that these methods may be taught in VM courses, but not necessarily called as such.

The similarities of various techniques between the two systems further shows that Tirtha's VM seems to be following the lineage of early Indian mathematics. Given the similarities between the two systems for different fundamental arithmetic operations, one can expect similar comparisons in square-root, cubing etc. - subject of future publication.

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