THE THIRD DIAGONAL

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Introduction

The concept of the third diagonal of a cyclic quadrilateral was floated by an Indian mathematician, Narayan Pandit (14thC), (hereafter referred to as NP), in his text Ganit Kaumudi (hereafter referred to as GK). The topic, third diagonal, is discussed in the fourth chapter of GK.

This paper is intended to explain the original concept of the third diagonal, the derivations of some results in reference to third diagonal (not proved in GK) and some of its applications.

1. Narayan Pandit

Narayan Pandit (NP) was born in Uttar Pradesh, India before about 1340, but no definite information is available. He has been titled as "Pandit" (learned) because of his intelligence. His father Narsinha was a well-known astrologer.

He wrote three texts: *Ganit Kaumudi. Bijganita Vatamsa and Karma Pradipika.* The last one contains NP's comments on the text *Lilavati* by Bhaskaracharya (12thC). All these texts are in Sanskrit.

2. Ganit Kaumudi

The most significant text after those of Bhaskaracharya is Ganit Kaumudi (GK). The topics covered in GK include weights and measures, partnership, triangles, quadrilaterals (both cyclic and non-cyclic), their constructions, their areas using third diagonal, shadows, elementary algebra and divisibility in numbers.

GK also contains partitions, fractions, combinatronics, areas, altitudes, series and sequences, arithmetic and geometric progressions, explaining binomial coefficients using a triangle, called as *Khand-Meru* (truncated triangle) algebraic equations of first and second degree using numerical examples, etc. Chapters 4 to 9 are specifically devoted to geometry.

NP and *Shridhara* (13thC) were the first two mathematicians to discuss trapeziums. Whenever necessary, a diagrammatic approach is also given. An earlier mathematician, named *Aryabhat* (5thC), described several results in algebra using a geometrical treatment, in his text *Aryabhatiya*.

NP was the first mathematician who dealt with magic squares, magic circles, magic hexagons, etc. NP called these squares as *Bhadra* or *(pious)* squares. They are included in the last chapter of GK.

GK gives a detailed treatment of geometry. Interestingly, there is also a descriptiom of arithmetic progressions, represented by isosceles trapeziums.

3. The Third Diagonal

A Sanskrit verse from GK, [1], page176, gives the definition of the third diagonal.

सर्वचतुर्बाहूनां मुखस्य परिवर्तने यदा विहिते ।
कर्णस्तदा तृतीयः पर इति कर्णत्रयं भवति ॥

When the top-side and the flank-side of a cyclic four sided figure (a cyclic quadrilateral) are interchanged, a third diagonal is generated.

Consider □$ABCD$ inscribed in a circle of radius r. (Fig 1 b)

Now interchange the sides AB and BC. This will create a point B' on the circumference of the circle such that arc BC = arc AB' and arc AB = arc $B'C$. Draw $B'D$, the diagonal of □ $AB'CD$.

This $B'D$ is termed as the third diagonal of □$ABCD$ in addition to two original diagonals AC and BD.

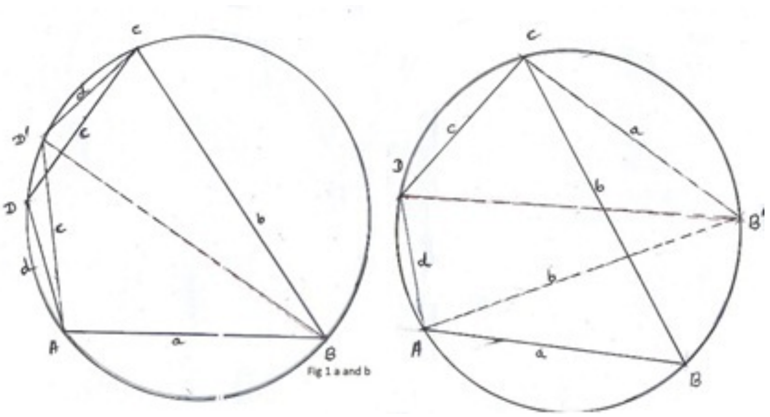

Fig 1a Fig 1b

As shown in Figure 1a, another third diagonal can be generated by interchanging the sides CD and DA, that is, by constructing arc CD = arc AD' and arc DA = arc CD' .

Here, we get a point D' on the circumference of the circle and a new □ $ABCD'$ is generated. Draw diagonal BD' of this quadrilateral. This segment BD' is also the third diagonal of □$ABCD$, in addition to two original diagonals AC and BD.

How many such “third diagonals” can be generated?

The third diagonals can be generated only by interchanging the adjacent sides of a quadrilateral (and not the opposite sides). Hence,

(i) Interchanging AB, BC will generate a third diagonal $B'D$ and a □ $AB'CD$, (fig 1b).

(ii) Interchanging BC, CD will generate a third diagonal $C'A$ and a □ $ABC'D$, (fig 2b)

(iii) Interchanging CD, DA will generate a third diagonal BD' and a □ $ABCD'$, (Fig1a)

(iv) Interchanging DA, AB will give a third diagonal $A'C$ and a□ $A'BCD$, (Fig2a)

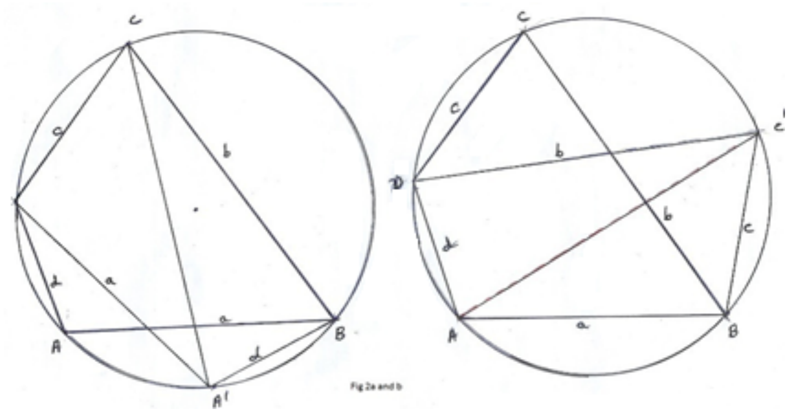

Fig 2a   Fig 2b

So only four third diagonals can be generated and these will generate four new quadrilaterals.

Note that, (i) Areas of all the five quadrilaterals, one original and four newly constructed quadrilaterals, are equal. This means,

A(□ABCD) = A(□ $A'BCD$) = A(□ $AB'CD$) = A(□ $ABC'D$) = A(□ $ABCD'$)

(ii) All the third diagonals are equal in length, i.e. $CA' = BD' = DB' = AC'$.

4. Some basic results

(i) (*A)* Area of triangle is equal to the products of its sides divided by four times its circumradius.

To prove: Area of a $\Delta ABC = \dfrac{abc}{4r}$, where r is radius of acircumcircle of ΔABC and a, b and c are the lengths of its sides**.**

Proof: Draw AM perpendicular from vertex A to base BC. Also let AD be the diameter through A.

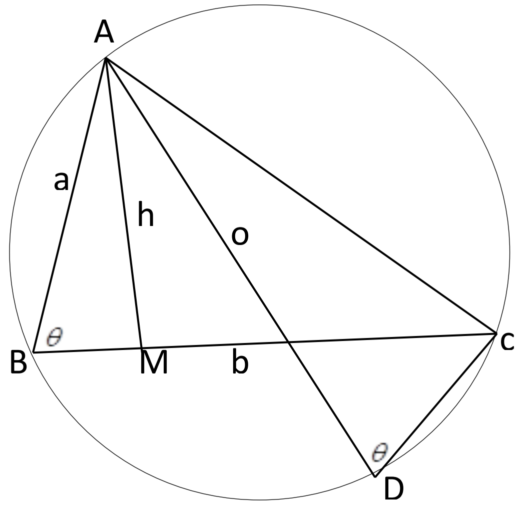

fig.3

Now $A\hat{B}C = A\hat{D}C = \theta$, because of angles on the same arc *AC*.

Also, $A\hat{M}B = A\hat{C}D$ (each right angle)

Hence, in $\Delta\, AMB$, $\sin\theta = \dfrac{AM}{AB} = \dfrac{AM}{a}$

Also, in ΔACD, $\sin\theta = \dfrac{AC}{AD} = \dfrac{c}{2r}$

Equating these two results, the altitude of triangle, $AM = \dfrac{ac}{2r}$

Therefore, Area $\Delta ABC \;=\; \dfrac{1}{2}BC.AM \;=\; \dfrac{1}{2}BC.\dfrac{ac}{2r} \;=\; \dfrac{abc}{4r}$

(ii)(B) Product of diagonals

In a cyclic □ *ABCD* with sides *a*, *b*, *c*, *d* anddiagonals *e* and *f*,

To prove that $ef = ac + bd$.

Dwigunavyasvibhakte, trikarnghatethavaganitanm [1], page 176.

Meaning**:** *The sum of products of opposite sides of acyclic quadrilateral is equal to product of its diagonals.*

This result is known *as Ptolemy's theorem,*

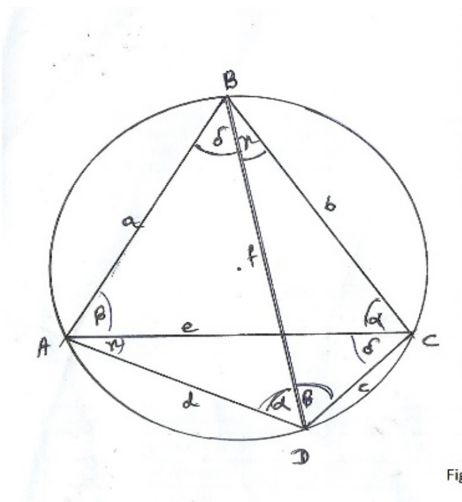

Fig 4a

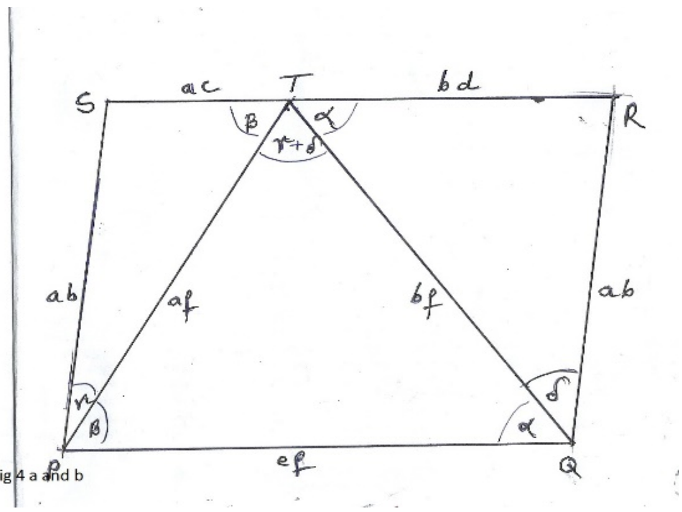

Fig 4b

Proof:

Construct Δ*PTQ*, Δ*PTS*, and Δ*TQR* (Fig 4b) similar to Δ*ABC*, Δ*BDC* and Δ*DBA* (Fig 4a) respectively, such that the ratio of their corresponding sides is *f*, *a* and *b*, respectively.

So, $\frac{PQ}{AC} = f$ and hence $PQ = AC.f = ef$.

Similarly the other ratios will give $QR = a.b$, $TR = b.d$, $QT = b.f$, $PS = a.b$, $ST = ac$, and $PT = a.f$

Arrange these triangles, taking into consideration, the sides of same length. This arrangement will make figure *PQRS* a parallelogram.

Now, length $ST = a.c$ and length $TR = b.d$, so that $SR = ST + TR = a.c + b.d$

Since □*PQRS* is a parallelogram, then $SR = PQ = e.f$

Equating, $e.f = a.c + b.d$

(iii)(C): Ratio of diagonals

In a cyclic □*ABCD*, let a, b, c, and d, denote its sides and e and f are its diagonals, then,

$$\frac{e}{f} = \frac{ad+bc}{ab+cd}$$

Let $AC = e$ and $BD = f$

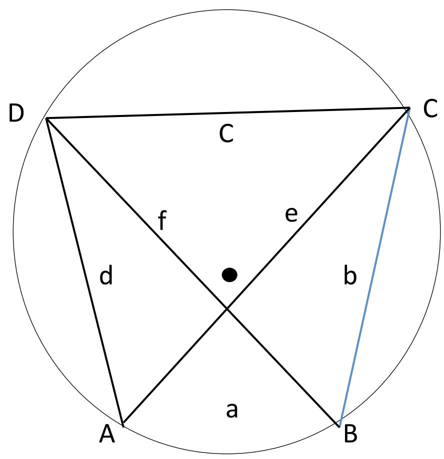

Fig 5

Proof:

Area □*ABCD* = Area Δ*ABC* + Area Δ*ACD* – triangles formed by diagonal *AC*

$$= \frac{eab}{4r} + \frac{ecd}{4r} = \frac{e(ab+cd)}{4r}$$

Area □*ABCD* = Area Δ*ABD* + Area Δ*BCD* – triangles formed by diagonal *BD*

$$= \frac{fad}{4r} + \frac{fbc}{4r} = \frac{f(ad+bc)}{4r}$$ (See theorem 1a)

Equating the two similar results, we have,

$$\frac{e(ab+cd)}{4r} = \frac{f(ad+bc)}{4r} \quad \text{or} \quad \frac{e}{f} = \frac{ad+bc}{ab+cd}$$

5. Applications

5.1. Area of a cyclic quadrilateral is equal to product of three diagonals divided four times circumradius**.**

To prove: Area $\square ABCD = \frac{AC.BC.AC'}{4r}$,

where $\square ABCD$ is a cyclic quadrilateral with diagonals AC and BD, AC' its third diagonal and r is the radius of the circumcircle.

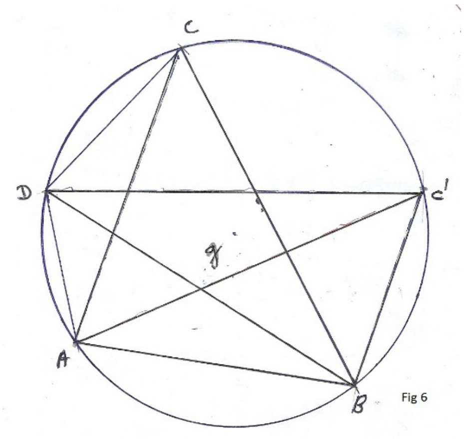

Fig .6

Proof:

Referring to fig 6, we have,

Area $\square ABCD$ = Area ΔACD + Area ΔABC

$$= \frac{AD.DC.AC}{4r} + \frac{AB.BC.AC}{4r} = \frac{AC(DC.AD + AB.BC)}{4r}$$

Now interchanging BC and CD, and creating a third diagonal AC', we get $DC = BC'$ and $BC = DC'$.

Hence, Area $\square ABCD = \frac{AC(BC'.AD + AB.DC')}{4r}$,

and by Brahma Gupta`s theorem (now Ptolemy`s theorem), the sum of products of opposite sides is equal to the product of diagonals, (result (2) above)

$$BC'.AD + AB.DC' = AC'.BD$$

Hence, Area $\square$ABCD $= \frac{AC.AC'.BD}{4r}$

Thus, area of a cyclic quadrilateral is equal to the product of its three diagonals divided by 4 times the circumradius.

5.2 *To prove ;*

$$\frac{g}{f} = \frac{AD.AB + BC'.DC'}{AB.BC' + AD.C'D} \text{ where } g = AC' \text{ and } f = DB.$$

Proof:

With referrence to fig. 7

Consider quadrilateral $ABC'D$ and apply theorem (iii).

We get the above result directly.

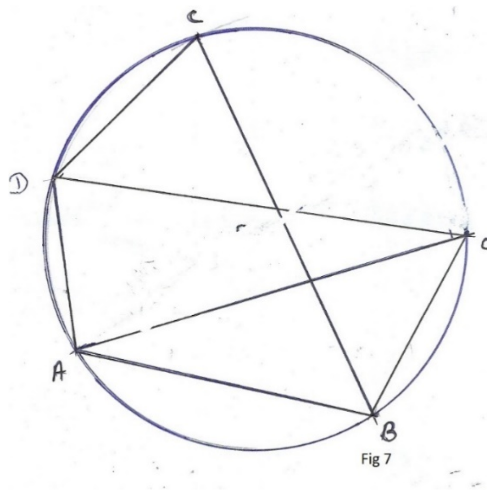

Fig 7

5.3 To find diagonal in terms of diameter.

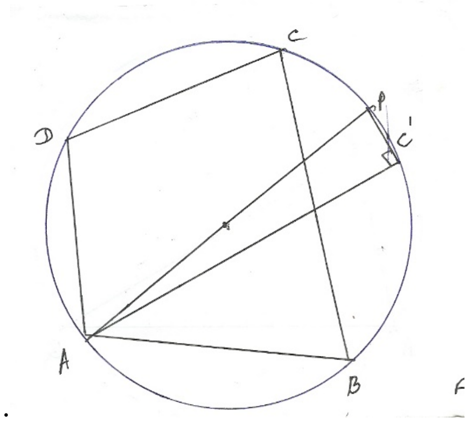

Fig.8

Through point *A*, draw a diameter *AP* and join P to C'.

We have $AP^2 = AC'^2 + PC'^2$

Hence, $AC'^2 = AP'^2 - PC'^2$ or $AC'^2 = d^2 - PC'^2$, where *d* is diameter of circumcircle.

Similarly, we can find the square of lengths of all the other three third diagonals.

5.4 Sections of intersecting diagonals

Let E be the point of intersection of diagonals AC and BD of a cyclic quadrilateral $ABCD$.

Let $AB = a$, $BC = b$, $CD = c$ and $DA = d$. Let $AE = i$, $EC = j$, $BE = k$, $ED = l$, so that $e = i + j$ and $f = k + 1$.

The diagonals are $AC = e$ and $BD = f$, and the third diagonal $AC' = g$.

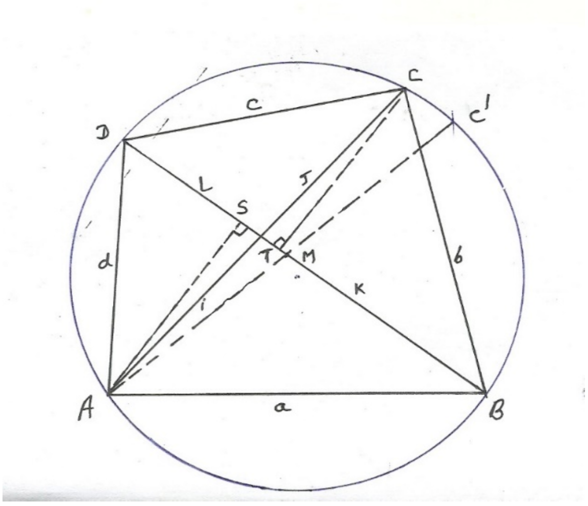

Fig 9

To prove:

(a) (i) $i = \dfrac{ead}{ad + bc}$, (ii) $j = \dfrac{bce}{ad + bc}$, (iii) $k = \dfrac{abf}{ab + cd}$ (iv) $l = \dfrac{cdf}{ab + cd}$

(b) (i) $i = \dfrac{ad}{g}$, (ii) $j = \dfrac{bc}{g}$, (iii) $k = \dfrac{ab}{g}$, (iv) $l = \dfrac{cd}{g}$

refer fig.9

(a) Proof:

Referring to fig. 9, the perpendiculars from A and C join BD at S and T respectively.

Now, ΔASE and ΔCTE are similar.

Hence, $\dfrac{AS}{CT} = \dfrac{l}{j}$ and $2r = \dfrac{bc}{CT} = \dfrac{ad}{AS}$

Hence, $\dfrac{AS}{CT} = \dfrac{ad}{bc} = \dfrac{i}{j}$ from the above.

Therefore, $\dfrac{i}{ad} = \dfrac{j}{bc} = \dfrac{l + j}{ad = bc} = \dfrac{e}{ad + bc}$

This gives, finally, $i = \dfrac{ead}{ad + bc}$.

Similarly $j=\dfrac{bce}{ad+bc}$, $k=\dfrac{abf}{ab+cd}$, and $l=\dfrac{cdf}{ab+cd}$ can be proved.

(B)Proof:(see figure 9)

Using theorem (C), $\square$ABCD, $a.c + b.d = e.f$. Similarly, for $\square\ ABC'D$,
$BC'.AD+AB.C'D=f.g$, but, because of interchange of adjacent sides, $BC'=CD=c$ and $C'D=BC=b$, hence in $\square\ ABC'D$, $a.b + c.d = f.g$

We have, from (A) above, $k=\dfrac{a.b.f}{(a.b+c.d)}$

Putting the value of $a.b + c.d$, $k=\dfrac{a.b.f}{f.g}=\dfrac{a.b}{g}$, as required.

Similarly we can show the other results:

$i=\dfrac{a.d}{g}$, $j=\dfrac{b.c}{g}$ and $l=\dfrac{c.d}{g}$.

5.5 Let E be the point of intersection of diagonals AC and BD, and F be the point of intersection of diagonals AC' and BD. Let CP, BQ and AR be the diameters through C, B and A. Segment AC' is the third diagonal.

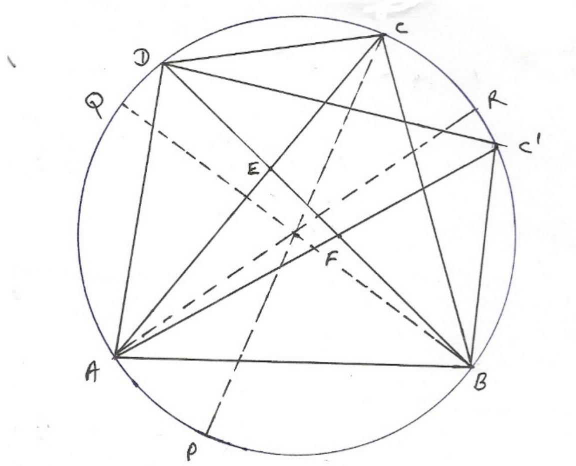

Fig 10

Then, $AE \sim CE$ = (AP.C\`R +DQ.C\`R).d / AC\`,

BE ~ DE = (AP.C\`R --DQ.C\`R).d / AC\`,

Also, $AF \sim C\`E = (AP.DQ + AP.C\`R).d\,/\,AC,$

$DF \sim B\`F = (AP.DQ -- AP.C\`R).d\,/\,AC,$

This gives the differences between the segments of diagonals (proof omitted).

5.6 In triangle $AB`C$, $AB`/\sin(ACB`) = 2r$.

Hence, length of third diagonal = $AB` = 2r\sin(ACB`)$.

Interestingly, the lengths of all third diagonals are proportional to sine of opposite angles made with original diagonal.

5.7 *To show:* $AB.BC + CD.DA = AB`.BD$

Proof: From result 4(B) above, we have, in □ $ABB`D$,

$AB.B`D + BB`.DA = AB`.BD$,

but by interchange of sides, $B`D = BC$ and $BB` = CD$,

Putting these values in above result, we get, (See figure 9)

$AB.BC + CD.DA = AB`.BD$, as required.

5.8 Length of diagonals in terms of sides

To show **:**(i) $AC = e = [(ac + bd)(ab + cd) / (ab + cd)]^{1/2}$

(ii) $BD = f = [(ac + bd)(ad + bc) / (ab + cd)]^{1/2}$

(iii) $AC` = g = [(ab + cd)(ad + bc) / (ac + bd)]^{1/2}$

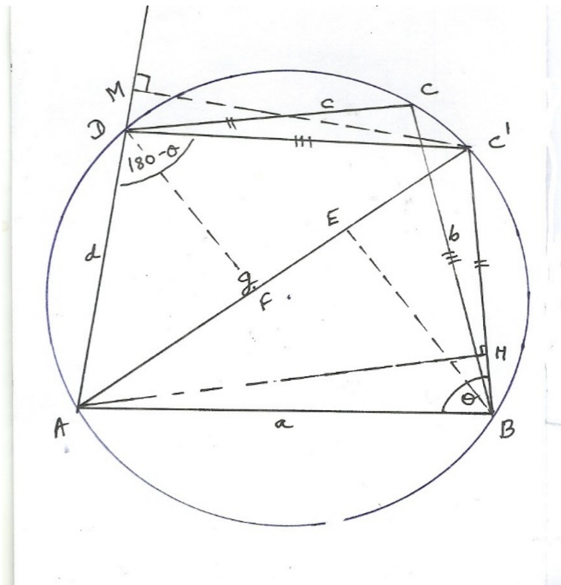

Fig 11

*Proof***:** Here, we will prove only third result, which is for third diagonal and the rest two results will follow by similarity.

Draw AH perpendicular from A to $BC`$ and $C`M$ perpendicular from $C`$ to AD (extend AD if necessary).

In Δ $AHC`$, $g^2 = AC`^2 = AH^2 + HC`^2$,

$= AH^2 + [HB^2 + B`C^2 - 2HB.BC`]$

$= [AH^2 + HB^{2]} + BC`^2 – 2HB.BC`$

$= AB^2 + B`C^2 -2(AB.\cos\theta).BC`$

But $BC` = CD = c$ and $C`D = BC = b$

Hence, $g^2 = a^2 + c^2 - 2.ac.\cos\theta$ …………………….. (1)

Similarly, in Δ $AMC`$, expanding as above,

$$g^2 = AC`^2 = AM^2 + MC`^2,$$

$$= b^2 + d^2 – 2bd \cos(\pi – \theta)$$

$$= b^2 + d^2 + 2bd \cos \theta \qquad …………………….. (2)$$

Multiply (1) by bd and (2) by ac and adding,

$$(ac + bd)g^2 = (a^2 + c^2)bd + (b^2 + d^2)ac = (ab + cd)(bc + ad)$$

i.e. $g^2 = [(ab + cd)(bc + ad)] / (ac + bd)$

or, $AC` = g = \{[(ab + cd)(bc + ad)] / (ac + bd)\}^{½}$,

Also other diagonals $AC = e$ and $BD = f$ can be obtained by similar constructions.

In particular, if, $b = c = d = a$, $ABCD$ will be a cyclic square and then

$C`$ will fall on $C$, the third diagonal $AC`$ will coincide with AC. Therefore all the diagonals will be equal, each of length $a\sqrt{2}$.

5.9 Area $\square ABC`D$**:**

Draw perpendiculars DN and BM on $AC`$. (ref fig. 12)

Let $DN = p$ and $BM = q$, $AC` = g$,

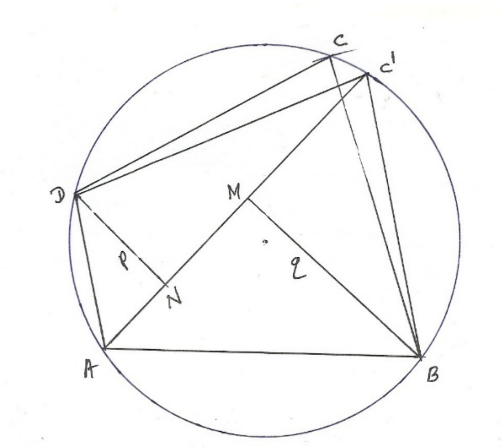

Fig 12

then, $\square ABC`D = \Delta ABC` + \Delta ADC`$

$$= (1/2)\, p.g + (1/2)\, q.g = (1/2)\,(p + q).g\ .$$

5.10 To prove: $p = 2.Ai / a.e$ and $q = 2A.k/a.f$,

where A is area of □$ABCD$, $i = AM$, k = BM,

p = perpendicular DE on AB,

q = perpendicular CF on AB,

$f = BD$, $e = AC$, $g = AC$`,

r is circumradius and d is circumdiameter (Proof omitted)

Corollary: $p = f.i.g / a.D$ and $q = e.k.g / aD$

The proof follows by putting A = area(□ $ABC`D$)= $e.f.g / 2D$ in value of p from above.

5.11 To find the circumradius.

A mathematician, *Parmeshwar*, from Kerala state, (15 century) had already obtained the formula for the radius r of a circle circumscribing the quadrilateral $ABCD$.

Here, the same result is obtained by using the third diagonal.

Interchange the sides AD and CD, so that new position of D is D`. And then $CD$` = AD and AD` = $DC$. $BD$` is a third diagonal of $ABCD$.

The new quadrilateral $ACD`D$ is a cyclic isosceles trapezium with $DD$` parallel to AC and AC parallel to the diameter *x-o-x'*

Draw BK perpendicular to extended DD`. Then $BK$ is the altitude of Δ $BDD$`.

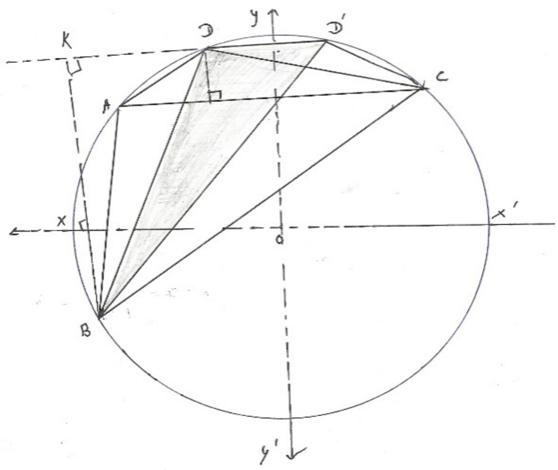

Fig 13

Hence, $BK = (BD.BD`) / 2r$ …………………….. (1)

Also , BK is sum of altitudes of Δs ABC and ADC.

Therefore, Area(□ $ABCD$) = Area (Δ ABC) + Area (Δ ADC)

= (1/2) AC (sum of altitudes)

= (1/2) AC. BK

Hence, BK =2 Area (□ $ABCD$)/ AC ……………………..(2)

Equating (1) and (2), $r = BD.BD`.AC / 4(\text{Area} (\square ABCD))$

or, $r = (BD.\ BD`.AC)\ / 4 \sqrt{[(s - a)(s - b)(s - c)(s - d)]}$

where a, b, c, d, are the sides of $\square ABCD$, and $2s = a + b + c + d$.

Now, Product of three diagonals = $BD.BD'.AC$

$$= \sqrt{(a.b + c.d)(a.c + b.d)(bc + bd)}$$

$(s - a)(s - b)(s - c)(s - d) = (a + b + c - d)(a + b - c + d)(a - b + c + d)(-a + b + c + d)$

Hence,

$r = \{(ab + cd)(ac + bd)(bc + ad) /$

$(a + b + c - d)(a + b - c + d)(a - b + c + d)(-a + b + c + d)] \}^{1/2}$

5.12 The sum of products of adjacent sides about a diagonal of a cyclic quadrilateral is equal to the product of that diagonal and third diagonal**.**

That is, to prove**:** $AB.BC + CD.DA = BD.B`D$

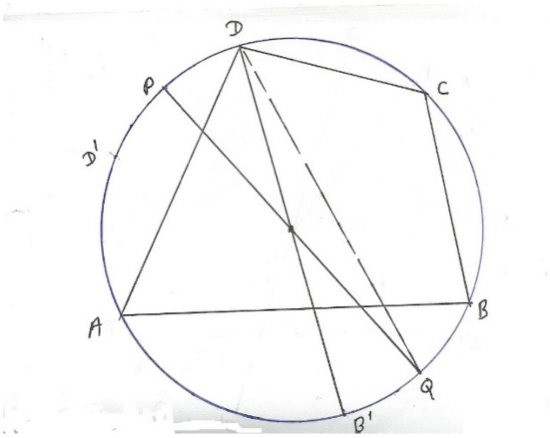

Fig 14

Proof: In quadrilateral $ABCD$, let $AB > BC > CD > DA$

Interchange sides AB and BC, so that a new point $B`$ is created such that $AB` = BC$ and $B`C = AB$. This generates a third diagonal $B`D$.

Take a point $D`$ on the circle such $AD` = DC$.

Let P and Q be the midpoints of arcs $DD`$ and $B`B$ respectively.

Now, arc $QB`$ = arc $QB$ and arc $DP$ = arc $D`P$

Therefore, arc PAQ = arc $PD`$ + arc $D`A$ + arc $AB`$ + arc $B`Q$

= arc PD + arc DC + arc BC + arc BQ

= arc PC + arc CQ = arc PCQ

This gives that PQ is a diameter.

We have, arc AD.arc CD = arcs$[(AD + CD) / 2]^2$ -- $[(AD - CD) / 2]^2$,

$$= AP^2 - PD^2 \text{..................... (i)}$$

Similarly, arcAB .arc BC = $AQ^2 - BQ^2$ ....(ii)

Adding the results (i) and (ii), and using that $AP^2 + AQ^2 = PQ^2$, as APQ is a right angle triangle, we have,

$$AB.BC + CD.DA = PQ^2 - PD^2 - QB^2$$

$$= QD^2 - QB^2$$

$$= (QD + QB)(QD - QB)$$

$$= (QD + QB).(DQ - QB)$$

$$= BD.\ B`D$$

5.13 Another proof of Ptolemy`s theorem involving third diagonal:

To prove: Product of diagonals of a cyclic is equal to the sum of product of its opposite sides.

Proof: interchange the sides BC and CD, so that BC = DB` and DC = BB`. This generates a third diagonal *AB*`.

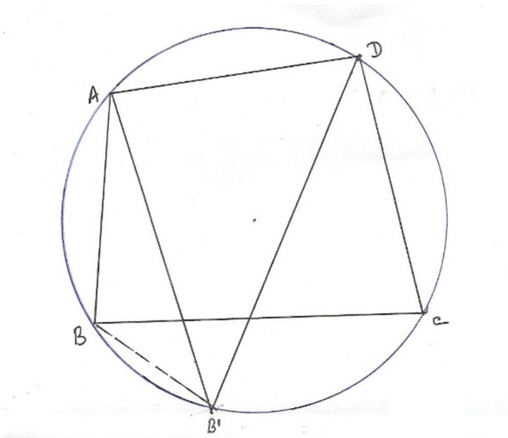

Fig 15

Using result of 5.12 above for diagonal *BD*, *AD.DC* + *AB.BC* = *BD.AB*`

Now replace *DC* by *BB*` and *BC* by *DB*`, so that, *AD.BB*` + *AB.DB*` = *BD.AB*`

This proves Ptolemy`s theorem because for quadrilateral, *ABB`D*,

(*AD*, *BB*`) and (*AB*, *DB*`) are the pairs of opposite sides and (*BD*, *AB*`) is a pair of its diagonals.

Conclusion: The results given above all contain a third diagonal.

The specialty of this paper is that their proofs are easier when we use the third diagonal.

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Note:

The authors do not claim any originality in this paper, except some explanations.