A New Approach to the Teaching of Coordinate Geometry

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Abstract

The Vedic Mathematics sutras of Shankaracarya Sri Bharati Krishna Tirtha can be usefully employed in the teaching and learning of coordinate geometry in two ways, described in two parts of this paper. In the first instance, elementary coordinate geometry is used to illustrate how the sutras appear in conventional treatment of a subject. This is due to the fact that the sutras express fundamental ideas, thought-patterns, cognitions and concepts that underlie the forms, strategies and problem-solving mechanisms that are in common use. The simple applications within coordinate geometry also help to reveal the meaning of the sutras. By use of the sutras, common formulae of various types take on a slightly different form in order to facilitate better understanding and greater efficiency. They furnish us with fast techniques for solving problems that students face in their public exams.

The second part of the paper takes one aphorism, or formula, from Vedic Mathematics and applies it to various aspects for finding areas of shapes in the context of coordinate geometry. The formula deals with areas of parallelograms and triangles. It then provides a spectacular one-line proof of Pythagoras' theorem. Further applications reveal how it also gives simple proofs of compound angle formulae used in trigonometry and a short and easy proof of the formula used for finding the distance from a point to a line. It is believed that these proofs, Pythagoras' theorem, etc., have not been published before now.

Keywords: Vedic Maths, Coordinate Geometry, Pythagoras' Theorem, Compound Angle Formulae

Part 1

Modern analytic geometry seems to have begun by both Renee Descatres (1596 – 1650) and Pierre Du Fermat (1607 – 1665). Although the use of axes placed orthogonally goes back to the ancient Greeks, it was Descartes and Fermat who discovered that a line can be used to represent a relation between two variables and vice versa.

A.E.Young reports that Analytic Geometry is a dependent Science because it is the unification of algebra and geometry. It is no wonder then that when analytic geometry is first taught to children there can easily arise blockages of understanding where there is insufficient prerequisite knowledge of geometry or algebra.

This paper investigates how the sutras relate to graphs in very simple ways; giving a new approach to how coordinate geometry can be taught.

1.1 Basic Straight Line

A straight-line graph, such as $y = 0.8x$ (Figure 1), expresses a proportionality between the two variables. The sutra is Anurupyena – Proportionately.

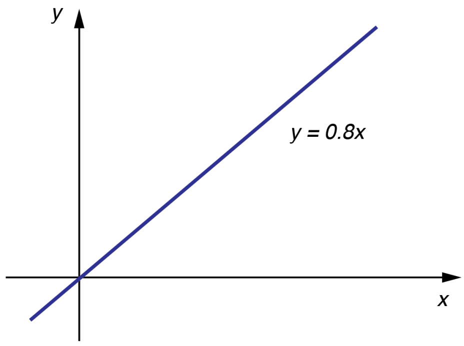

Figure 1

There are many practical applications of this such as a Distance/Time graph (Figure 2). For a constant speed, distance is proportional to time elapsed. The constant of proportionality is the speed, shown as the gradient of the line.

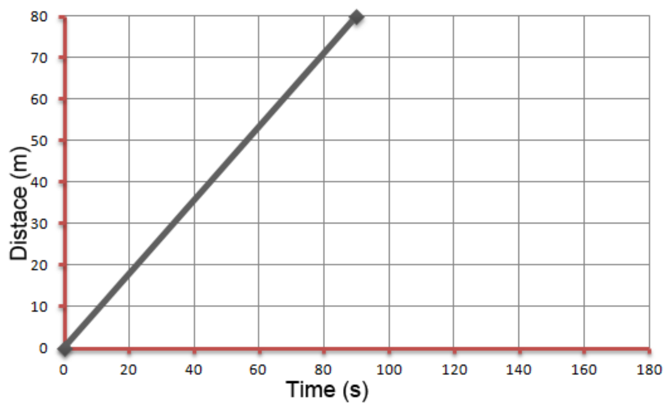

Figure 2

The straight line in Figure 1 can be translated to a new position (Figure 3) and the equation is altered into the familiar $y = mx + c$ mode. Together with Proportionately this translation is expressed by Paravartya Yojayet – Transpose and Adjust.

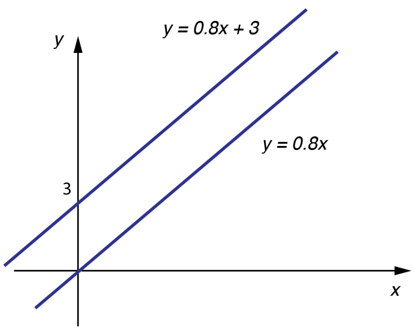

Figure 3

1.2 How to Draw a Line from its Equation

Suppose we wish to draw the line whose equation is $2x - 3y = -12$ (Figure 4)

A simple application of the Lopanasthapanabhyam sutra – By Elimination and Retention, reveals the intercepts with both axes.

When $x = 0, y = 4$
When $y = 0, x = -6$

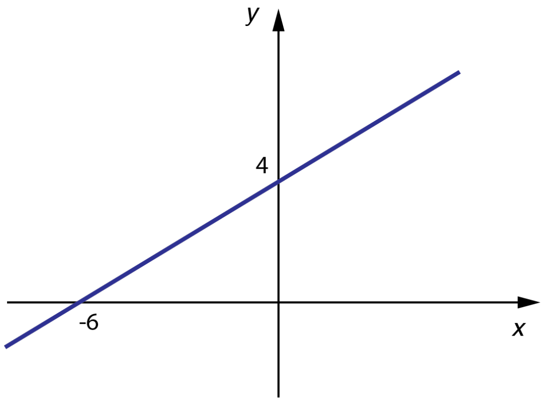

Figure 4

1.3 How to Find the Midpoint of a Line-Segment

Example Find the midpoint of the line-segment joining the points (2, 3) and (10, 7).

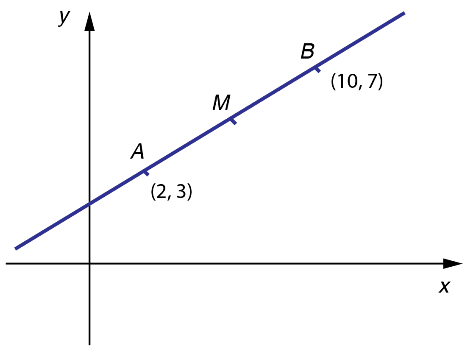

Figure 5

This can be achieved by taking the average of the *x* and *y* coordinates.

$x = \dfrac{2+10}{2} = 6$ and $y = \dfrac{3+7}{2} = 5$. So *M* lies at (6, 5).

The relevant sutra is Vyashti Samashti – Specific and General.

The sutras mentioned so far have myriads of applications because they express simple and fundamental concepts that occur very frequently.

1.4 How to tell if a point is on a given line

Example Does the point (9, 3) lie on the line with equation $2x+3y=27$?

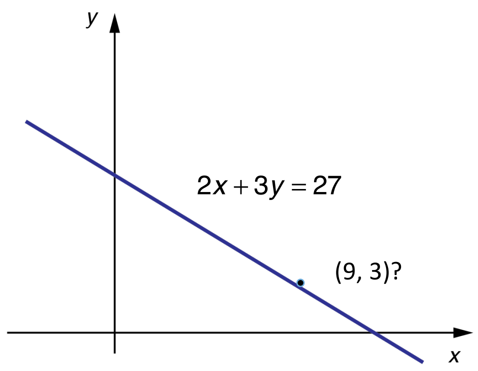

Figure 6

By substituting the values into the left-hand side of the equation we can see if the total is the same as the right-hand side.

$$2\times 9 \quad + \quad 3\times 3 \quad = \quad 18+9 \quad = \quad 27 \quad \text{Yes!}$$

The sutras involved here are Transpose and Apply and Sunyam Samyasamuccaye – When the total is the same it is nought.

In his book, His Holiness Bharati Krishna Tirtha, gave two translations for Paravartya Yojayet, Transpose and Adjust and Transpose and Apply. It is the most frequently quoted sutra in his text. In this context, the simple substitution of values into an equation is one application. The Sunyam sutra is not so easy to understand, actually because of its simplicity. The central concept is that when two aggregates are identical then the difference between them is nothing. This is a very fundamental concept in mathematics.

1.5 How to Find the Equation of a Line Given its Gradient and One Point that Lies on the Line

The current and conventional approach to this very common problem in coordinate geometry is cumbersome. It either uses the equation $y=mx+c$ and substitutes in values for m, together with the particular coordinates in order to find the value of c or uses the formula,

$$y-y_1=m(x-x_1)$$

This formula inevitably requires some manipulation to get the equation into a satisfactory shape.

The Vedic one-line formula is not only more effective but also reveals an important concept.

The formula is $mx-y=mx_1-y_1$ and is based on the Specific and General sutra.

Example Find the equation of the line passing through the point (7, 5) and with gradient 2.

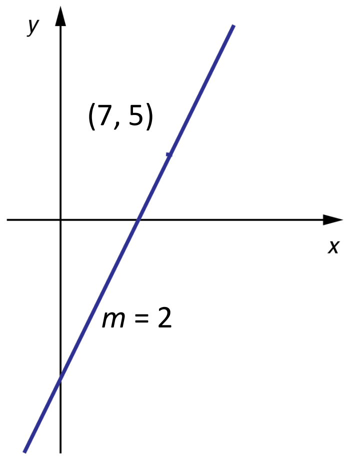

Figure 7

Substituting the values into the equation gives, $2x - y = 9$.

There are four advantages to this approach over and above the conventional. Firstly, the equation can be established in one line of mental working. Secondly, it is in a form that is frequently required for further work, such as solving simultaneous equations. Thirdly it strongly reflects that one side of the equation is general and the other side is specific. Fourthly, it is easy to remember because the two sides of the formula mirror each other.

The conventional formula, $y - y_1 = m(x - x_1)$, does have one advantage in that it can be seen as a transformation of the line $y = mx$ by replacing y with $y - y_1$ and x with $x - x_1$ and this would come under the Transpose and Adjust rule.

1.6 Parallel Lines

With the $mx - y = mx_1 - y_1$ format, the left-hand side of parallel lines remains unaltered; it is only the independent term that changes. The sutra is Transpose and Adjust.

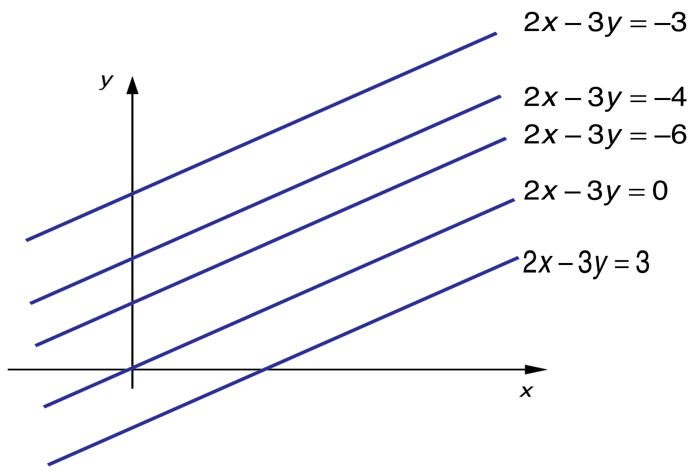

Figure 8

1.7 How to find the equation of a line parallel to a given line that passes through a given point

The left-hand side remains the same and the independent term on the right is evaluated by inserting the coordinates of the known point into the left-hand side.

Example Find the equation of the line parallel to the line $3x + 5y = 17$ that passes through the point (2, 1).

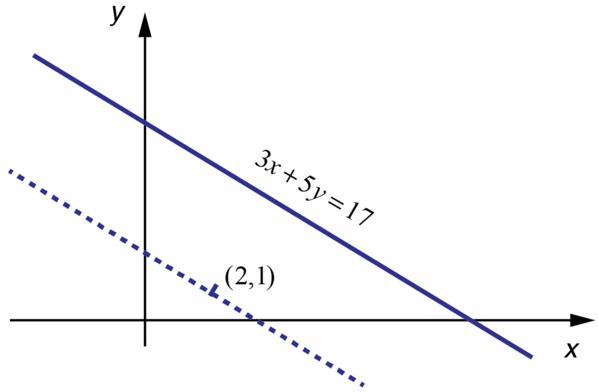

Figure 9

The left-hand side is the same. Evaluate the right-hand side by inserting known values into the left-hand side.

$$3x+5y = 3\times 2 + 5\times 1$$
$$3x+5y = 11$$

1.8 Perpendicular Lines

With the equation in the form $ax - by = c$ the gradient is $\frac{a}{b}$ in which case a perpendicular has gradient $-\frac{b}{a}$. The result is that a perpendicular is of the form $bx + ay = d$. In other words, the left-hand side of the equation of a perpendicular can be found by transposing the x and y coefficients and changing the sign between. As before, the right-hand side is found by evaluating the left-hand side using the values of a known point.

Example Find the equation of the line perpendicular to the line $3x + 5y = 17$ that passes through the point (2, 1).

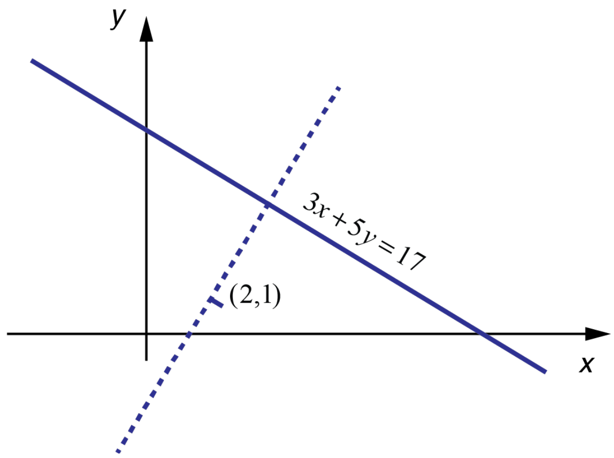

Figure 10

Transposing the left-hand side gives $5x-3y$ and evaluating the right-hand side gives $5\times2-3\times1=7$. The required equation is therefore $5x-3y=7$. This is a double application of Transpose and Adjust.

1.9 Equation of a Line Through Two Given Points

This is another common problem in coordinate geometry and one which, in fact, Tirthaji describes in his book. The equation is of the form $ax-by=c$ and all that is required is to find the value of the three constants, *a*, *b* and *c*, using the coordinates of the two points.

The value of *a* is the difference of the two *y*-coordinates and the value of *b* is the difference of the two *x*-coordinates. These need to be subtracted in the same direction.

The right-hand side of the equation is found by using the rule, Product of the Means minus Product of the Extremes, as is shown below.

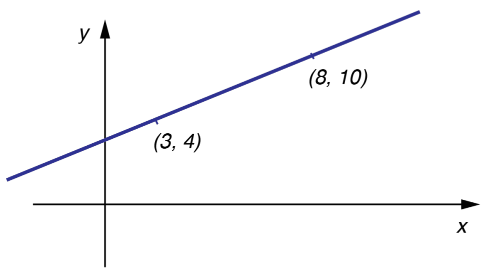

Figure 11

For ease, the larger numbers are written first, $(8,10)\ (3,4)$.

$a=10-4=6$ and $b=8-3=5$. The left-hand side is then $6x-5y$.

The right-hand side is $10\times3-8\times4=-2$ and the equation is therefore $6x-5y=-2$.

This can be achieved in one line of mental working.

One conventional method is as follows:

$$m=\frac{10-4}{8-3}=\frac{6}{5}$$
$$y-4=\frac{6}{5}(x-3)$$
$$5y-20=6(x-3)$$
$$5y-20=6x-18$$
$$-20+18=6x-5y$$
$$6x-5y=-2$$

Another conventional method is to use the formula,

$$\frac{y-y_1}{y_2-y_1}=\frac{x-x_1}{x_2-x_1}$$

This has some merits inasmuch as it has symmetry but, unfortunately, still requires quite a lot of manipulation.

$$\frac{y-4}{10-4}=\frac{x-3}{8-3}$$

$$\frac{y-4}{6}=\frac{x-3}{5}$$

$$5(y-4)=6(x-3)$$

$$5y-20=6x-18$$

$$6x-5y=-2$$

Concluding remarks

The Vedic Maths sutras operate within very many simple operations.

It is possible to use such elementary topics to gain further understanding of what the sutras mean. The guiding question is, "When I am performing mathematical tasks what is the sutra at work?" Sometimes it is more than one sutra.

The Vedic methods also provide efficient ways to solve problems.

Part 2

Here we will look at an application of the Vedic Maths formula "Product of the means minus product of the extremes" in connection with areas of parallelograms. The aphorism provides some surprising results together with a one-line proof of Pythagoras' Theorem and applications to trigonometry.

2.1 Area of a Parallelogram

Figure 12 shows a parallelogram with one vertex on the origin and two adjacent vertices at (4, 5) and (6, 3). The problem is to find the area.

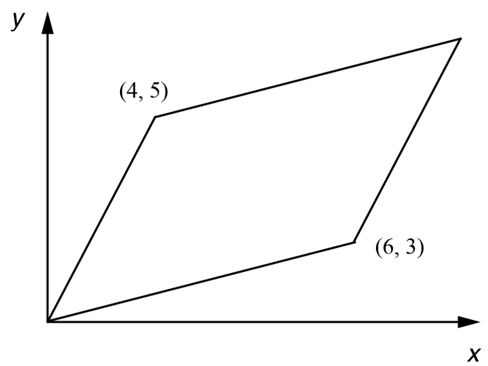

Figure 12

The rule requires setting out the coordinates in a row, (4, 5) (6, 3).

Product of the means minus product of extremes yields, $5\times6-4\times3=18$.

It is the positive value that is used; so if the coordinates are written the other way, (6, 3) (4, 5), -18 is the result and so the positive value is used.

If the parallelogram does not have one vertex on the origin then it can be translated by adjusting the coordinates.

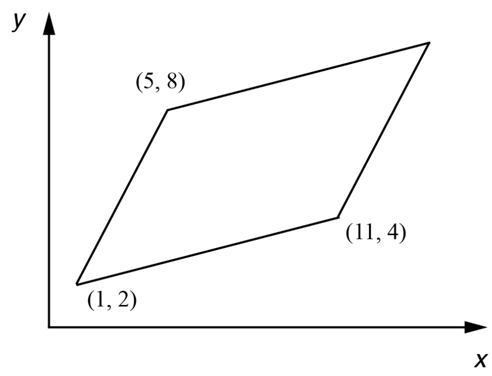

Figure 13

In this example, subtracting 1 from each of the other *x*-coordinates and 2 from the corresponding *y*-coordinates the parallelogram is repositioned so that one vertex lies on the origin.

$$\begin{array}{cc} (5,\ 8) & (11,\ 4) \\ \underline{(1,\ 2)} & \underline{(1,\ \ 2)} \\ (4,\ 6) & (10,\ 2) \end{array}$$

$$\text{Area} = 6\times10 - 4\times2 = 38$$

Triangles can be dealt with in a similar way since every triangle is half a parallelogram.

In the example below,

$$\text{Area} = \tfrac{1}{2}(5\times6 - 4\times3) = 9$$

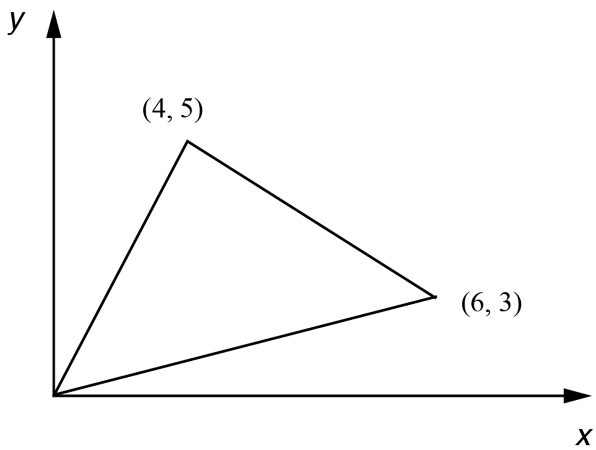

Figure 14

2.2 How does the Formula Work?

The Product of the means minus the product of the extremes is the same as the determinate of a transformation matrix and the determinate gives the area scale factor. When the transformation matrix is applied to a unit square then the determinate gives the area of the parallelogram and the columns in the matrix are the coordinates of the two vertices adjacent to the origin.

Another explanation of the formula can be seen in the following figures in which a parallelogram is dissected and transformed into a shape that is the difference of two rectangles.

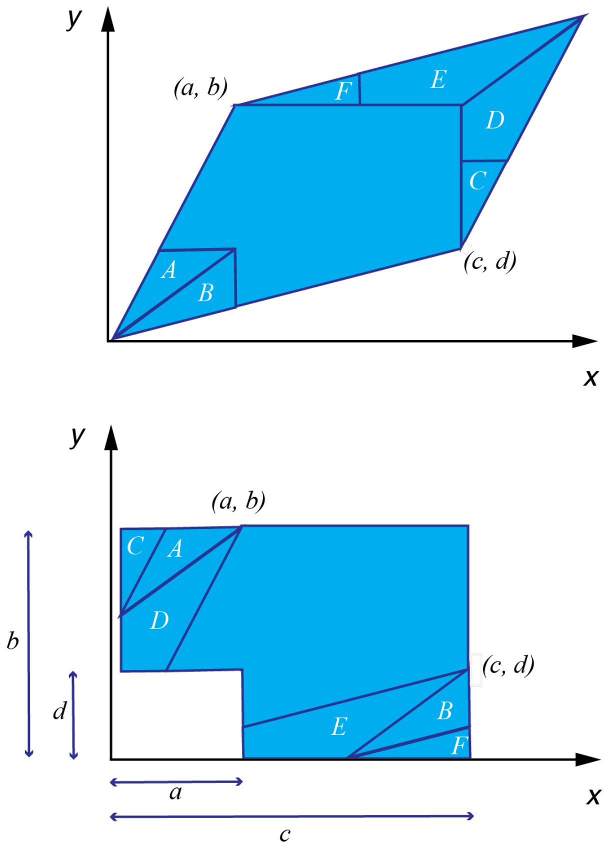

Figure 15

The area is found by subtracting the small vacant rectangle from the large, $bc - ad$.

There are other ways of showing this formula works. A simple method is to draw a rectangle around the parallelogram and then subtract trapeziums and triangle from it until the parallelogram remains.

2.3 Pythagoras' Theorem

This formula can be used for a one-line proof of Pythagoras' Theorem.

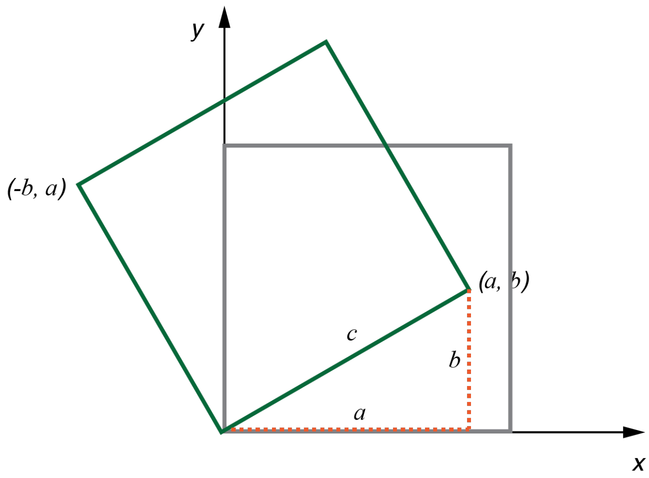

Figure 16

A square of side length c is rotated about the origin. If one adjacent vertex has position (a, b) then the other adjacent vertex has position (-b, a) as shown in figure 12.

Since a square is a parallelogram the formula can be used to find its area.

$$\text{Area} = a^2 - (-b^2)$$

But since the square has side length c, and area c^2, then $a^2 + b^2 = c^2$.

2.4 Compound Angle Formulae

Applications of the same rule can be used to provide short proofs of compound angle formulae for sine and cosine. This is done by allowing the parallelograms to have side length of unity.

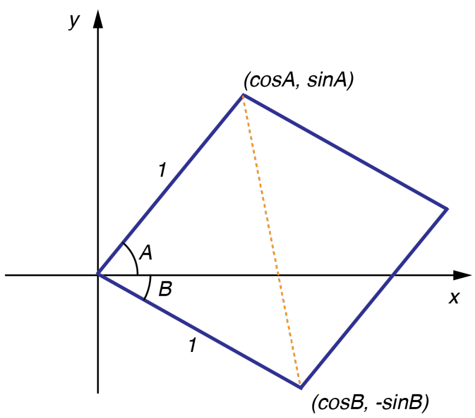

Figure 17

With edge length 1, the vertices of the parallelogram lie at $(\cos A, \sin A)$ and $(\cos B, -\sin B)$.

The area of half the parallelogram is then $\sin A \cos B - (-\cos A \sin B)$.

But from the area of a triangle formula, $A = \frac{1}{2}ab\sin C$, the area is also $\frac{1}{2}\times 1\times 1\times \sin(A+B)$.

Therefore $\sin(A+B) = \sin A\cos B + \cos A\sin B$.

The following figure is used for the negative case.

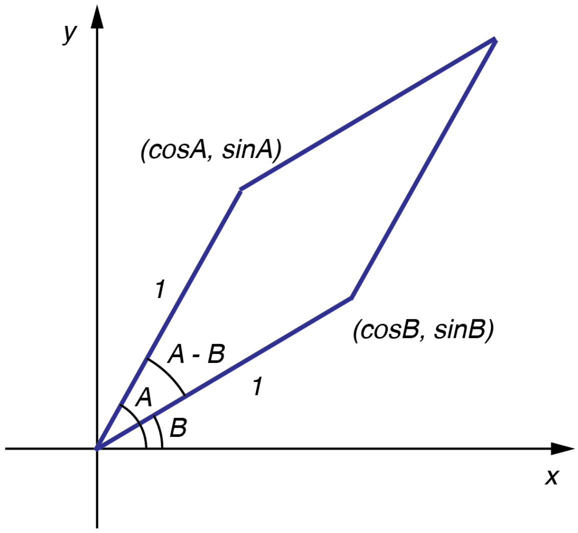

Figure 18

By the same procedure as before, $\sin(A-B) = \sin A\cos B - \cos A\sin B$.

For compound cosine the following figure is used:

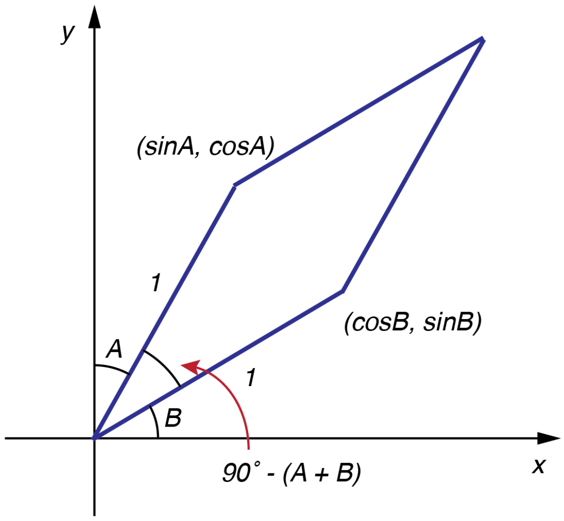

Figure 19

In this case the vertices of the parallelogram lie at $(\sin A, \cos A)$ and $(\cos B, \sin B)$ and the area of the parallelogram is then $\cos A\cos B - \sin A\sin B$.

The angle inside the parallelogram is $90^\circ - (A+B)$ and $\sin\left(90^\circ - (A+B)\right) = \cos(A+B)$.

Therefore, $\cos(A+B) = \cos A\cos B - \sin A\sin B$.

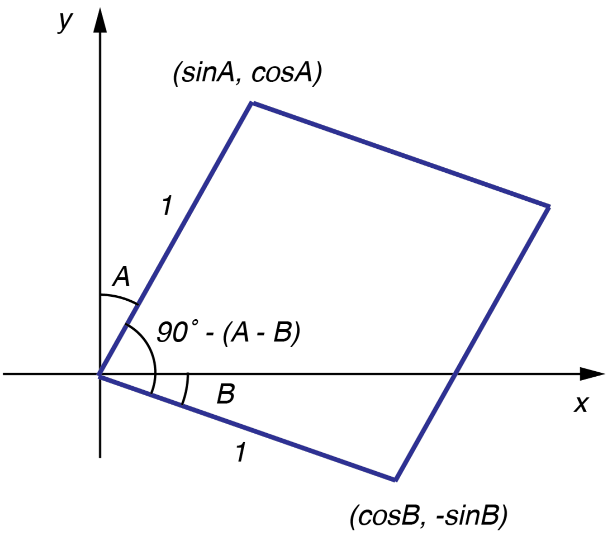

Figure 20

In figure 24, the required vertices lie at $(\sin A, \cos A)$ and $(\cos B, -\sin B)$.

The area is $\cos A \cos B + \sin A \sin B$.

The angle inside the parallelogram is $90^\circ - A + B$ which is the same as $90^\circ - (A - B)$. The sine of this angle is $\sin\left(90^\circ - (A - B)\right) = \cos(A - B)$.

Therefore, $\cos(A - B) \ = \ \cos A \cos B + \sin A \sin B$.

2.5 Distance from a Point to a Line

The formula for the area of a parallelogram is now used to derive the formula for the distance between a known point and a straight line.

The point P lies at (x_0, y_0) and the straight line has equation, $ax + by + c = 0$.

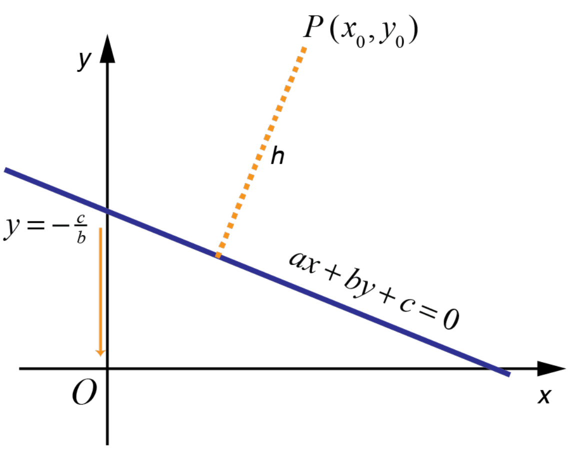

Figure 21a

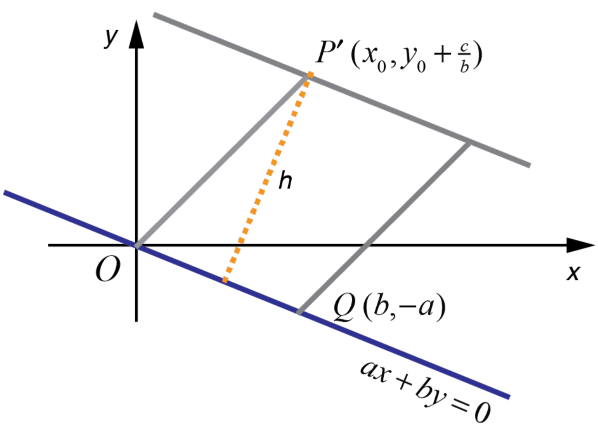

Figure 21b

The y-intercept of the straight line is $-\frac{c}{b}$. The point and line are translated downwards so that the line passes through the origin. As a result, P becomes $P'(x_0, y_0 + \frac{c}{b})$ and the line becomes $ax + by = 0$.

A point Q is chosen on the line with coordinates $(b, -a)$. This is possible since these values satisfy the equation of the transformed line. A parallelogram is then formed using sides OP' and OQ.

The area of the parallelogram is then $b\left(y_0 + \frac{c}{b}\right) + ax_0 = ax_0 + by_0 + c$

Using the Pythagoras' theorem the distance is $OQ = \sqrt{a^2 + b^2}$.

Since the area of a parallelogram is the product of the base, *OQ*, and its perpendicular height, h, it follows that,

$$h = \frac{ax_0 + by_0 + c}{\sqrt{a^2 + b^2}}$$

It is believed this derivation is more efficient that those provided elsewhere.

Concluding Remarks

In Part 1 of this paper are described common elementary aspects of coordinate geometry from the standpoint of the Vedic Mathematics sutras. It illustrates that the sutras express fundamental mathematical concepts frequently overlooked. It also shows that, in some cases, the "Vedic" approach gives greater efficiency as well as deeper understanding. Such is the case where the two sides of the equation of a line reflect the general and particular aspects. This concept, in turn, helps appreciation of the same at a later stage with differential equations.

Part 2 reveals a very simple formula that can be used to give very easy proofs of important results used in analytic geometry such as Pythagoras' theorem compound angle formulae and the distance from a point to a line. It is believed that these derivations have not been seen before.

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