VEDIC MATHEMATIC DEVICES FOR SQUARING

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Abstract

Not only does each of the Vedic Maths sutras have multifarious applications but also many individual processes have various techniques, each governed by a different sutra. This is a hallmark of the incredible flexibility of the system. This paper explores various methods of squaring using as many of the sixteen sutras as possible. There are two approaches to this investigation. The first is to see how the Vedic methods are applied to squaring and the second is to take existing methods and see which sutras apply. The result is a wide range of techniques, some of which are universal whilst others depend on particular circumstances and produce special-case formulae.

This paper also discusses the benefits of mult-approach solutions to problems.

1. By one more than the one before

This is used for squaring numbers ending with 5.

Example 1 35^2

The digit before the 5 is 3 and one more than this is 4. Multiply $3 \times 4 = 12$ to give the first part of the answer. The second part of the answer is the square of 5.

$$35^2 = 1225$$

This can be extended with reference to place value:

$350^2 = 122500$, in which the number of zeros are doubled.

$3.5^2 = 12.25$

$0.035^2 = 0.001225$, in which the number of decimal digits is doubled.

How does it work?

In the case of 35^2 the following diagram illustrates the structure:

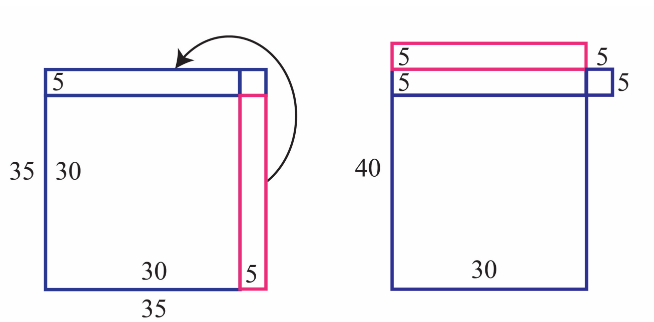

Figure 1

Let the first digit be x. A number such as 35 can be expressed algebraically as $10x+5$ (Figure 2).

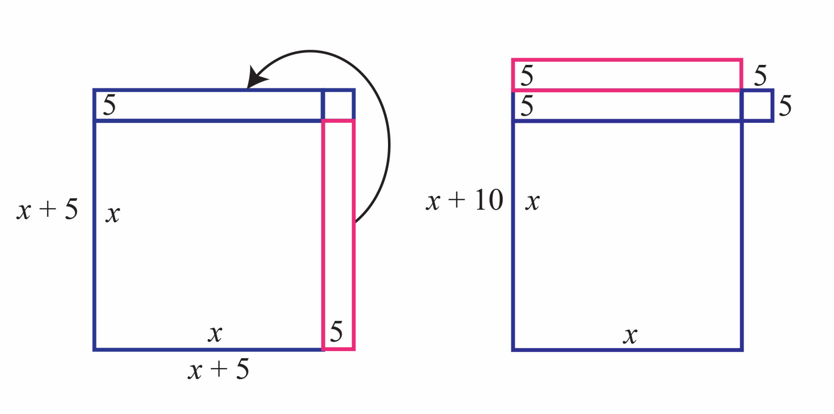

Figure 2

$$(10x+5)^2 = 100x^2 + 100x + 25 = 100x(x+1) + 25$$

x is the original digit and this is multiplied by one more than itself, $x+1$.

Under the same sutra but not connected to the above is the following pattern involving square numbers as the sum of consecutive odd integers.

$$1 = 1$$

$$1+3 = 4$$

$$1+3+5 = 9$$

$$1+3+5+7 = 16$$

$$1+3+5+7+9 = 25\ldots$$

Given an integer, n, $2n$ must be even and so $2n+1$ is odd.

The pattern above can be expressed as $\sum 2n+1 = n^2$ and this can be proven using the formula for the sum of an arithmetic series, $S_n = \frac{n}{2}\{2a+(n-1)d\}$, with $a=1$ and $d=2$.

2. Squaring Numbers Close to a Power of 10

This comes under the *All from 9 and the last from 10* sutra in conjunction with the sub-sutra, *Whatever the extent of the deficiency, lessen it still further, and set up the square of the deficiency,* which itself is an extension of the *Yavadunam* sutra.

Example 2 97^2

The deficiency is 3 and 97 decreased by 3 gives 94. The square of the deficiency is 09.

$$97^2 = 9409$$

A zero is needed to take account of place value.

The same method applies for numbers above a power of 10 but instead of a subtracted deficiency there is an added surplus.

Example 3 106^2

The surplus of 6 is added to 106 to give 112. The square of the surplus is 36.

$$106^2 = 11236$$

How does this work?

In the case of a number below the power of 10, let x be the deficiency.

$$(100-x)^2 = 10000 - 200x + x^2 \;=\; 100(100-2x) + x^2$$
$$97^2 = (100-3)^2 = 10000 - 200\times 3 + 3^2 = 100(100-2\times 3) + 9 = 100\times 94 + 9 = 9409$$

This is based on the algebraic identity, $(a\pm x)^2 \equiv a^2 \pm 2ax + x^2$.

3. Using a working base

This is an extension of the previous method and allows the number to be related to either a multiple of factor of a power of 10 using the sub-sutra, *Proportionately*.

Example 4 47^2

The working base, 50, is related to the real base 100 as $50 = 100 \div 2$.

$$50 = 100 \div 2$$
$$47 - 3 = 44$$
$$44 \div 2 = 22$$
$$3^2 = 09$$
$$47^2 = 2209$$

Example 5 23^2

$$20 = 10\times 2$$
$$23 + 3 = 26$$
$$26\times 2 = 52$$
$$3^2 = 9$$
$$23^2 = 529$$

Figure 3 refers to Example 5. The square is divided into five sections. The area of each large rectangle is 230 and that of each small rectangle is 30. In the calculation 23 + 3 = 26, but due to place value it is acutally 230 + 30 = 260 and this gives the area of one large and one small rectangle. This is then doubled to give 520. The small square with area 9 does not have to be doubled.

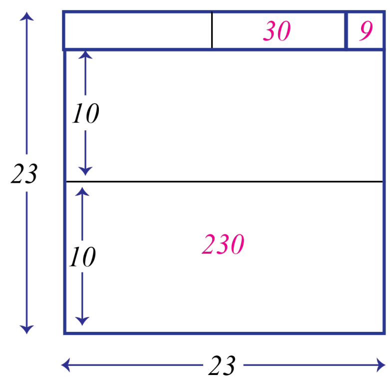

Figure 3

4. Vertically and Crosswise

The vertically and crosswise, straight squaring, method uses duplexes.

For a single digit number, *e*, the duplex is the square, e^2.

For a double-digit number, *de*, the duplex is double the product, $2\times d\times e$.

For a three-digit number, *cde*, the duplex is $2\times c\times e\ +\ d^2$.

For a four-digit number, *bcde*, the duplex is $2\times b\times e\ +\ 2\times c\times d$.

For a five-digit number, *abcde*, the duplex is $2\times a\times e\ +\ 2\times b\times d\ +\ c^2$, and so on.

Example 6 23^2

Step 1 Put down the duplex of 3, i.e. 9

Step 2 Put down the duplex of 23, i.e. $2\times 2\times 3={}_{1}2$

Step 3: Put down the duplex of 2, i.e. 4 and add the carry 1, giving $5_{1}29$.

This can be extended to squaring numbers of any size all in one line of working.

Example 7 423^2

$$\begin{array}{rl}
3 & D=3^2=9 \\
23 & D=2\times 3\times 2=12 \\
423 & D=4\times 3\times 2+2^2=28 \\
42 & D=4\times 2\times 2=16 \\
4 & D=4^2=16
\end{array}$$

$$423^2=\ 17_{1}8\ _{2}9\ _{1}2\,9$$

5. Transpose and Adjust

Squares can be found by transposing the number into an easy product and then adjusting by adding a small square.

This is based on the identity, $x^2-a^2\equiv(x+a)(x-a)$, that transposes to $x^2\equiv(x+a)(x-a)+a^2$

Example 8 58^2

Choosing 60 as the 'easy' multiplier, the deficiency, 2, is added and subtracted (sub-sutra, *By addition and subtraction*) to and from 58, giving 60 and 56. The square of the deficiency is then added.

$$\begin{aligned}58^2&=60\times 56+2^2\\&=3364\end{aligned}$$

6. By addition and subtraction

A closely related method also arises from the identity for the difference of two squares, but taken in a different way, namely, $a^2 - b^2 = (a+b)(a-b)$ in the form, $b^2 = a^2 - (a+b)(a-b)$.

Instead of an 'easy' multiplier it uses an 'easy' square.

Example 9 38^2

A close easy square is 40^2 .

$$\begin{aligned} 38^2 &= 40^2 - (40+38)(40-38) \\ &= 1600 - 78 \times 2 \;=\; 1600 - 156 \;=\; 1444 \end{aligned}$$

78 is the sum of 40 and 38 and 2 is the difference.

This is illustrated in Figure 4.

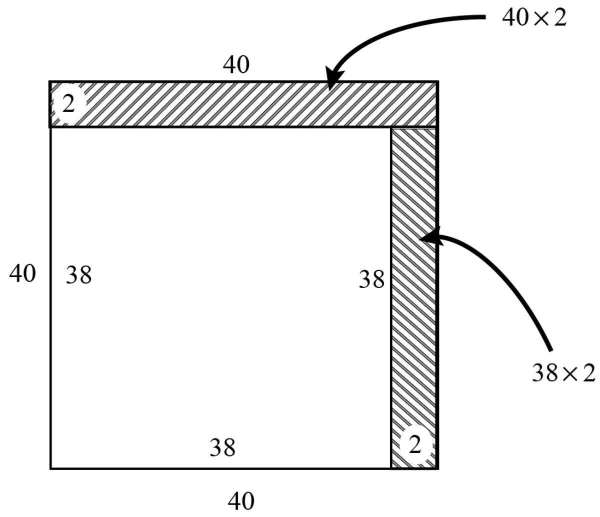

Figure 4

7. Completing the square

The sutras involved here are *By completion and non-completion* and *When the total is the same it is zero*.

Example 10 Solve, by completing the square, $x^2 + 8x + 16 = 0$.

On seeing that 16 is required to complete the square, this can be added to both sides giving,

$$\begin{aligned} x^2 + 8x + 16 + 11 &= 16 \\ x^2 + 8x + 16 &= 16 - 11 \\ (x+4)^2 &= 5 \\ x + 4 &= \pm\surd 5 \\ x &= -4 \pm \surd 5 \end{aligned}$$

The first three terms, $x^2+4x+16$, form the completed square and the remaining terms form the non-completed part.

In respect of *When the total is the same, it is zero*, one meaning of this expresses the balance that is retained with any manipulation of an equation and may be read as 'When the total on both sides of the equation is the same, the difference is zero'. This expresses the fundamental idea of equality. In the case above 16 is added to both sides and so the balance is retained.

8. Differential Calculus

Calculus can be used to raise a binomial to any power by successive differentiation and integration. In this case 2 is the relevant power. A two-digit number is a binomial in which each digit is treated as a "variable" that can be differentiated or integrated.

The general formula for this is written about in Reference 2 and is as follows:

$$(a+b)^n = \sum_{i=0}^{n} D_i(a^n) I_i(b^0)$$

where a and b are any numbers treated as variables, or just variables, $D_i(a^n)$ is the i^th^ differential of a^n and $I_i(b^0)$ is the i^th^ integral of b^0, from $i=0$ to $i=n$.

Example 11 72^2

Set down the first digit to the power of 2 and the last digit to the power of 0.

$$7^2 \times 2^0$$

The first term is differentiated and the second is integrated, $2\times 7^1 \times 2^1$

The first term of this result is again differentiated and the second term is integrated,

$$1\times 2\times 7^0 \times \frac{2^2}{2}$$

These three are placed in sequence and, on taking account of place value, summed up.

$$7^2\times 2^0 \quad / \quad 2\times 7^1\times 2^1 \quad / \quad 1\times 2\times 7^0\times\frac{2^2}{2}$$

$$= 49 / {}_{2}8 / 4 = 5184$$

Of course, this method is normally used for expanding binomials or raising two-digit numbers to higher powers and is shown here merely to illustrate that it can be applied.

9. Using a Geometric Sequence

Related to the calculus in the previous section and also the binomial theorem is a method using a geometric sequence and this uses the *Proportionately* sutra. The ratio in the geometric sequence is the ratio of the two digits in the number. The square of the first digit is set down and the ratio is then applied to produce the next two terms. The middle term is duplicated and the sum gives the answer. This is illustrated in Example 12 where the two digits have a ratio of 1:2.

Example 12 36^2

$$\begin{array}{ccc} 9 & 18 & 36 \\ & 18 & \\ \hline 12 & {}_{3}9 & {}_{3}6 \end{array}$$

This method was introduced by Tirtha for cubing two digit numbers very easily and the method can be extended to raising two-digit numbers to any power.

10. Specific and General

In geometry, any given rectangle can be transformed into a square of the same area. This is achieved by constructing the geometric mean of the two sides of the rectangle, which then forms the side of the required square. Since a mean is used, the relevant sutra is *Specific and general*.

Example 13 Suppose a rectangle measures 7 by 5 units. The task is to construct a square with area 35 square units.

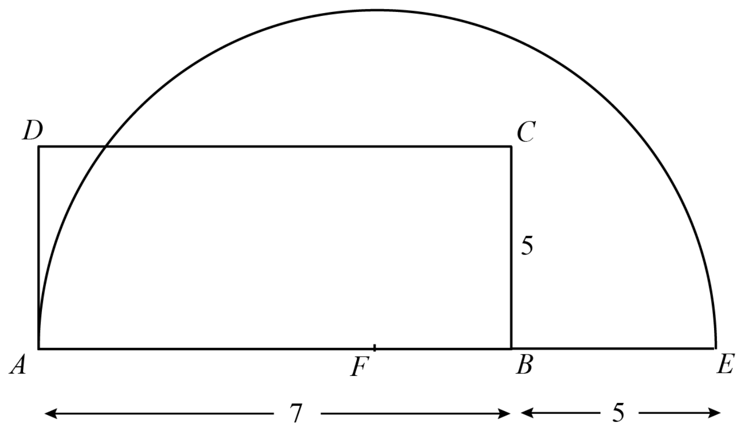

Figure 5

AB is produced to *E* so that *BE* = *BC*. *F* is the midpoint of *AE*. A circle is drawn with center *F* and radius *FE* (Figure 5)

BC is produced to cut the circle at *G* (Figure 6)

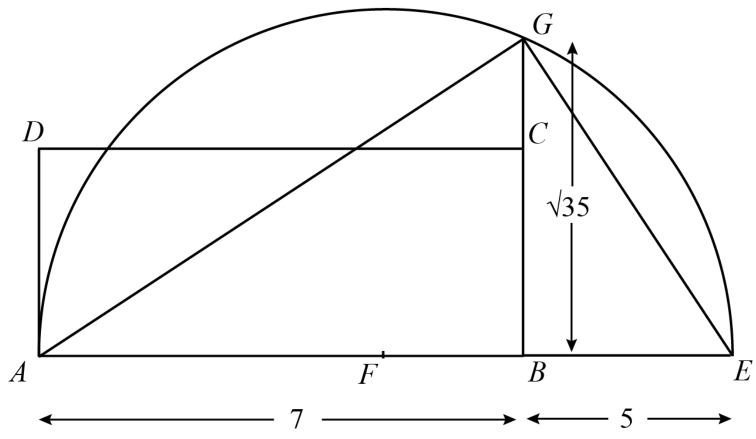

Figure 6

Triangles *ABG*, *GBE*, are similar with,

$$\frac{7}{BG} = \frac{BG}{5}$$

from which $BG = \sqrt{35}$. *BG* is the geometric mean of *AB* and *BE*.

A square is then drawn with side *BG* (Figure 7)

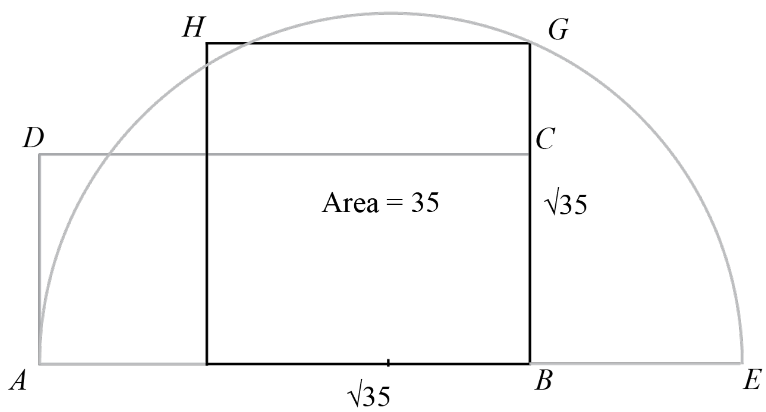

Figure 7

Use of the geometric mean can also be used to find the square root of a number geometrically. For example, $\sqrt{24}$ can be found be constructing a rectangle measuring 6 by 4 or 3 by 8, etc., factors of 24, and then producing the geometric mean.

Just as a rectangle can turn into a square so too can a triangle be transformed.

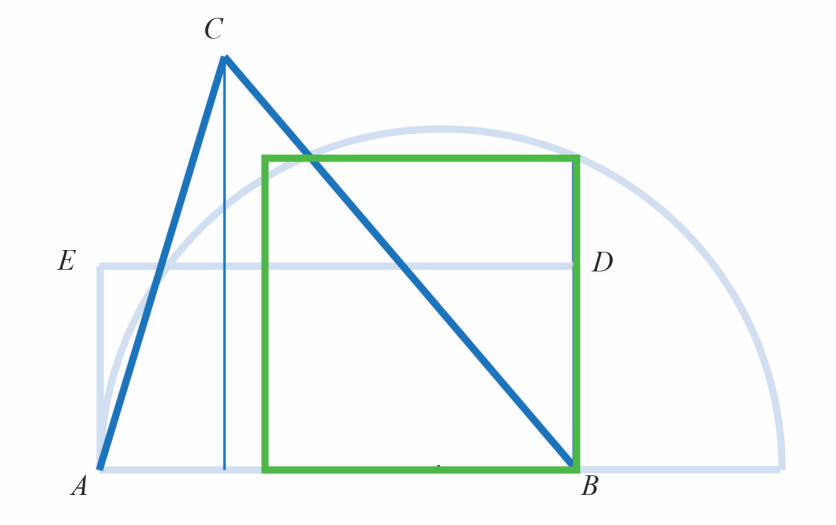

Figure 8

By halving the height of the triangle, a rectangle of equal area is created. This is then transformed into a square of equal area.

11. The Ultimate and Twice the Penultimate

This can be used where the number to be squared differs from a known square, or an easily obtainable square, by 1

The algebraic identity involved is $(x \pm 1)^2 \equiv x^2 \pm 2x + 1$.

Example 14 36^2

In this case, 35 is easily obtained from the *By one more than the one before* sutra as 1225.

$36^2 = (35+1)^2 = 1225 + 70 + 1 = 1296$

The ultimate is 1 and penultimate is 35. The ultimate and twice the penultimate are added to the square of 35 to give 1296.

Example 15 34^2

$34^2 = (35-1)^2 = 1225 - 70 + 1 = 1156$

In this case the calculation involves the ultimate and twice the negative penultimate.

Another application of this sutra is for squaring any number ending in 1. This is based on the same algebraic identity as above.

Example 16 $351^2 = 1225 / {}_{7}0 / 1 = 123201$

Care has to be given to the number of places provided for each part of the answer. In particular, the middle part of twice the penultimate is only allowed one digit and so a carry digit is often required.

With the use of vinculum digits this method can be used for squaring numbers ending in 9.

Example 17 $29^2 = 3\bar{1}^2 = 9\bar{6}1 = 841$

The ultimate is 1 (the square of $\bar{1}$). Twice the penultimate is $\bar{6}$ (the negative value is used because the ‘behind the scenes” calculation is $2 \times 3 \times \bar{1}$).

Example 18 $209^2 \ = \ 21\bar{1}^2 \ = \ 441/{}_{\bar{4}}\bar{2}/1 \ = \ 437/\bar{2}/1 \ = \ 43681$

This merely shows that it can be done although this is not the easiest method for many cases.

12. All the Multipliers

One simple device for squaring compound numbers is to multiply the squares of factors.

Example 19 18^2

$$18 = 2 \times 9$$

$$18^2 = 2^2 \times 9^2 = 4 \times 81 = 324$$

Example 20 27^2

$$27^2 = 3^2 \times 9^2 = 9 \times 81 = 729$$

13. Checking Answers

As with any arithmetic process answers can be checked using the Product of the sum sutra. For squaring this involves finding the square of the digital root of the number and comparing it with the digital root of the answer.

Example 21 Check that $4523^2 = 20457529$

The digital root of 4523 is 5 and the digital root of 5^2 is 7.

The digital root of 20457529 can most easily be found by casting out 9s and is 7.

It should be noted that this, of course, is not a full proof check. For example, if digits of the answer get interchanged then the digital root would still turn out to be 7.

Discussion

It is sometimes asked, of what benefit are all these different methods? In respect of education, these methods show that there is more than one way “to skin a cat”. Students who learn different techniques understand that different strategies can be used to solve a problem. This is of great use for training in problem solving. In the world of industry and commerce problem-solvers are highly employable.

Another benefit is that of breaking habit. Many children, and hence adults, learn blanket methods for simple calculations and stay within a limited mindset even though a different technique can be much easier. To be able to step outside habitual patterns of behaviour is always useful for human growth and development

Conclusion

It has been demonstrated that there are very many ways in which to square numbers. There are still more methods, such as grid multiplication, Russian multiplication, and so on, but since they may not live up to the Vedic ideal of quick as well as easy they have not been included here. Having stated that it should be noted that not all the methods described here are quick and easy. In some cases a method has been included simply to show the great flexibility of the Vedic Maths sutras and techniques.

In this exposition most of the sutras have been utilised. Of the sixteen, *those that have not been mentioned are, By one less than the one before, Remainders by the last digit* and *When one is in ratio the other is zero*. Some sub-sutras have also been used.

References

1. Tirtha B.K, *Vedic Mathematics*, 1965, Motilal Banarsidass

2. Fletcher M & Glover J, 2016, International Vedic Mathematics Conference, Kolkata, *How the Binomial Theorem Underlies the Working of the Anurupyena and Yavadunam Sutras in the Calculation of Successive Powers of a Number; Application of power Triangles and Calculus*