TEACHING CALCULUS

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Abstract:

Calculus comes under the Calana Kalanābhyām Sutra of Vedic Mathematics. Though it is a subject usually taught later in the school career, Sri Bharati Krishna Tirthaji tells us that in the Vedic system 'Calculus comes in at a very early stage'. This paper aims to show how Calculus may be taught to quite young children. Gradients of curves can be found without 'differentiation' in the usual sense, and areas under curves can be found without the usual integration process. Furthermore this approach uses a similar method for both thereby unifying these two aspects of Calculus. After the idea of a limit is introduced, in a simple way, and with a little algebra and geometry, we arrive at an easy technique that gives exact gradients of curves and areas under curves. There is no need for the complex notation that is normally associated with this subject and which adds to its perceived difficulty in school (though this notation can be introduced at the teacher's discretion).

Introduction

Calculus is usually taught towards the end of a school career because its sophisticated arguments require a good grounding in various other areas and its complex notations tend to baffle. Swami Tirtha makes it clear in the Prolegomena of his book[1] that the topics in Vedic mathematics can be taught in any order, and also that 'Calculus comes in at a very early stage' in the Vedic system.

Since Calculus is such an important and useful subject it makes sense that it should be introduced in a simple way early on. Other topics, like geometry, graph work, statistics and so on are introduced early on and developed from there. Why can't we do the same with Calculus? What we need is a simple approach that builds on what pupils know and keeps the notation within bounds.

This article shows that this can be done. If pupils know a little algebra and understand basic graphs and the idea of gradients, then they can take the approach to Calculus outlined here.

First we will look at how differentiation is approached traditionally before introducing the Vedic method. Also we will see how a simple relationship (that for $y = ax^n$ a certain ratio of gradients is equal to n) can also be used to get gradients. In the second section we see a certain ratio of areas that is also equal to n, and which can be used to easily obtain areas under curves. The area under the curves traditionally comes under integration in calculus. Thus the current paper covers both differentiation and integration, which form the core of the subject of calculus.

1. DIFFERENTIATION

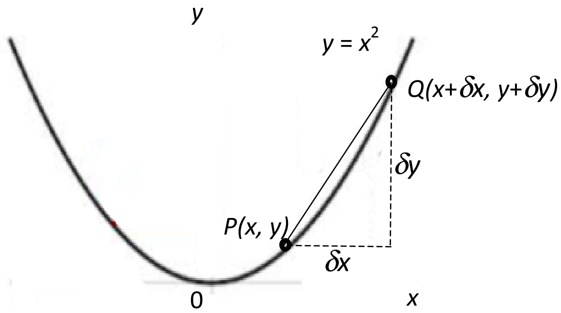
1.1 Modern Approach to Differentiation

The modern approach to teaching differentiation is to introduce function notation and define a tangent gradient of a curve by:

$$f(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

So in addition to the function notation we have the special notation for limits and the notion of vanishing quantities like δx, formed, confusingly, by two inseparable symbols.

Then a diagram shows the set-up for a particular case, invariably $y = x^2$.



Next the definition is applied, giving:

$$\begin{aligned} f'(x) &= \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x} \\ &= \lim_{\delta x \to 0} \frac{(x+\delta x)^2 - x^2}{\delta x} \\ &= \lim_{\delta x \to 0} \frac{2x\delta x + (\delta x)^2}{\delta x} \\ &= \lim_{\delta x \to 0} (2x + \delta x) \\ &= 2x \end{aligned}$$

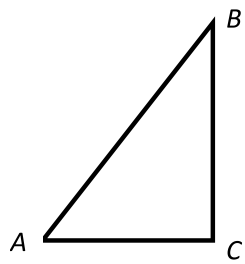
This may work well for the older, and more mathematically sophisticated child, and of course the definition is general and so applies to all functions, but the alternative approach given next is entirely suitable for younger children and introduces the concepts of limits and gradients much more simply.

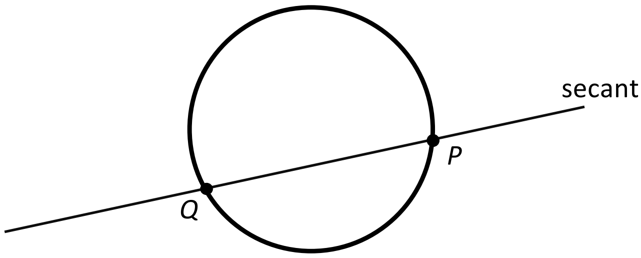
1.2 Vedic Approach to Differentiation

1.2.1 Limits

Children know about limits in their everyday life – they know there is a limit to the distance they can throw a ball or the height they will grow to.

From here we can give mathematical examples: like what is the limit of the line AB as B moves towards, and merges into, C:

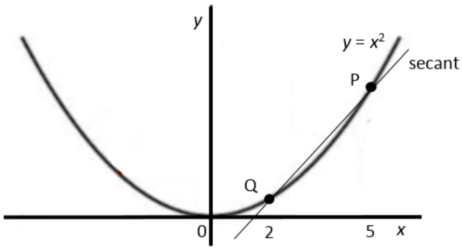




Or the limit of the secant line PQ as Q approaches P around the circle, and merges into P:

1.2.2 Secant of a Parabola

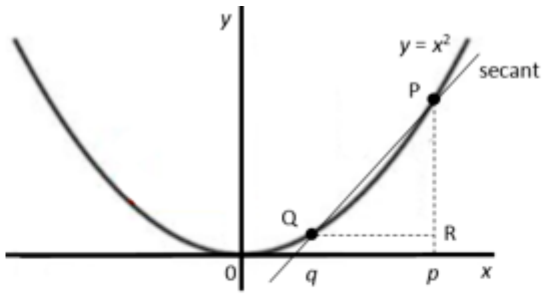
Suppose we have the simplest of mathematical curves: the parabola, described by the equation
$y = x^2$.



We can take two points on the curve, say where $x = 2$ and $x = 5$, and we can find the gradient of the secant that passes through them. Since the difference of the y-coordinates divided by the difference of the x-coordinates is $\frac{25-4}{5-2}$ we find the gradient to be 7.

Further examples lead to the observation of a simple relationship between the two x-coordinates: that the sum of those x-coordinates gives the gradient. In this case $2 + 5 = 7$, the gradient of the secant. We just use the Sutra *By Addition*.

We can go on to prove this result. Given points on the parabola with $x = p$ and $x = q$ we expect the gradient of the secant to be $p + q$.



We have: $P(p, p^2)$ and $Q(q, q^2)$ and
the gradient of $PQ = \frac{p^2 - q^2}{p - q} = \frac{(p-q)(p+q)}{p-q} = \boldsymbol{p + q}$.

This approach can then be developed to find secant gradients for:

- $y = ax^2$,
- $y = ax^2 + bx + c$,
- $y = x^3$ and so on.

The formula for the difference of two squares used above is well-known and very useful. It also leads to further formulae for the difference of two cubes etc., which can be made use of when finding secant gradients for other curves.

We will stay with $y = x^2$ for the moment.

1.2.3 Gradient of a Curve

We can discuss with pupils the idea of the gradient of a curve: that it varies, and that the gradient of the curve is the gradient of the tangent at that point.

Coming back to the secant through PQ, suppose we let Q approach P around the curve. Then the secant ultimately becomes the tangent at P and the value of q approaches the value of p, and ultimately becomes equal to p.

That means that since the secant gradient is $p + q$, the gradient at P is $p + p = 2p$, or $2x$ since p is an x-coordinate.

Pupils need to understand the generality and limit of this result: that it is true for all points on this particular parabola, but not necessarily true for any other curve. And they can confirm graphically that the gradient is always twice the x-coordinate.

This "secant-tangent method" (in which we find the secant gradient and then allow one of the points defining the secant to merge into the other) is extendible and can lead to the general result that for any polynomial the gradient formula can be found in the conventional way.

That is, that: $\frac{d}{dx}\sum_{i=0}^{i=n} a_i x^i = \sum_{i=0}^{i=n} a_i i x^{i-1}$.

In the last step of the derivation of $(p + q)$ for the secant gradient, the cancelling of $\frac{p-q}{p-q}$ is not valid when $p = q$, but this can be explained as appropriate in terms of limits.

This secant-tangent approach does not extend to other functions such as trigonometrical and exponential functions, but these can be tackled with alternative Vedic methods.

1.2.4 Extending

After studying the graphs of:

- $y = ax^2$,
- $y = ax^2 + bx + c$ (here we demonstrate the distributive law for differentiation holds),
- $y = ax^3$ etc.

in a similar way, we can look at negative indices:

- $y = \frac{1}{x}$,
- $y = \frac{a}{x}$,
- $y = \frac{a}{x^2}$ etc.

And then fractional indices:

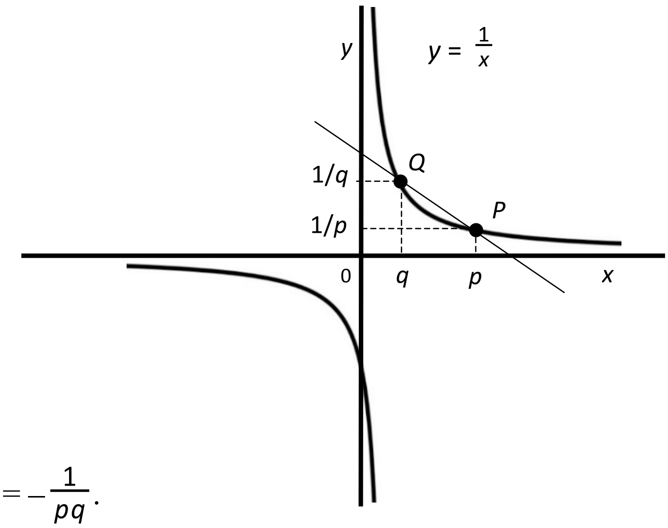
- e.g. $y^2 = 5x^3$,

which can lead into the chain rule, implicit differentiation and parametric equations.

All of these can be treated in the way described above.

Example

To find the gradient for $y = \frac{1}{x}$, for example, we find the difference of the y and x-coordinates for $P(p, \frac{1}{p})$ and $Q(q, \frac{1}{q})$.



The secant gradient is $\dfrac{\frac{1}{p} - \frac{1}{q}}{p - q} = -\dfrac{\frac{p-q}{pq}}{p - q} = -\dfrac{1}{pq}$.

Then in the limit, as $Q \to P$, the tangent gradient is $-\frac{1}{p^2} = -\frac{1}{x^2}$.

So simple curves like $y = x^2$, $y = \frac{1}{x}$ etc. have tremendous potential for teaching the fundamentals of differentiation – finding gradients of curves using a limiting process.

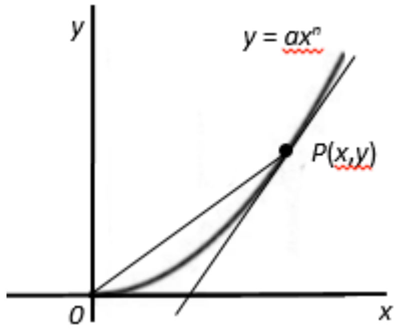
1.3 Ratio of Gradients

Here we have an alternative relationship that we will later connect with a similar ratio for areas under curves.

There are in fact two gradients associated with any point on a curve:

1) the gradient of the tangent to the curve at that point,

2) the gradient of the line joining the point to the origin.



And for $y = ax^n$, where n is rational and a is a real number:

The ratio of these gradients is *n*.

So we have: ratio of gradients $\frac{dy/dx}{y/x} = n$, which means that $\frac{dy}{dx} = n \times \frac{y}{x}$.

1.3.1 Proof that the Ratio of Gradients = *n*

Given $y^u = ax^v$, where u, $v\in$ ¢ , and $a\in$ ° , we have $uy^{u-1}\frac{dy}{dx} = avx^{v-1}$.

And since $\frac{y}{x} = \frac{ax^{v-1}}{y^{u-1}}$, then $\frac{dy}{dx} = \frac{v}{u} \times \frac{y}{x}$.

This result comes under the Sutra *If One is in Ratio the Other is Zero*. It can also be derived using the "secant-tangent method" given earlier. It

Example: Find the gradient of $y = 2x^3$ at the point (3, 54).

We note $n = 3$ and so: $\frac{dy}{dx} = 3 \times \frac{54}{3} = 54$.

Example: And to find the gradient of $y^2 = 4x^3$ at the point (4, 16), we note $n = \frac{3}{2}$ and so:

$\frac{dy}{dx} = \frac{3}{2} \times \frac{16}{4} = 6$.

For comparison with the contemporary method the corresponding solution to the last example above would be:

$2y\frac{dy}{dx} = 12x^2$

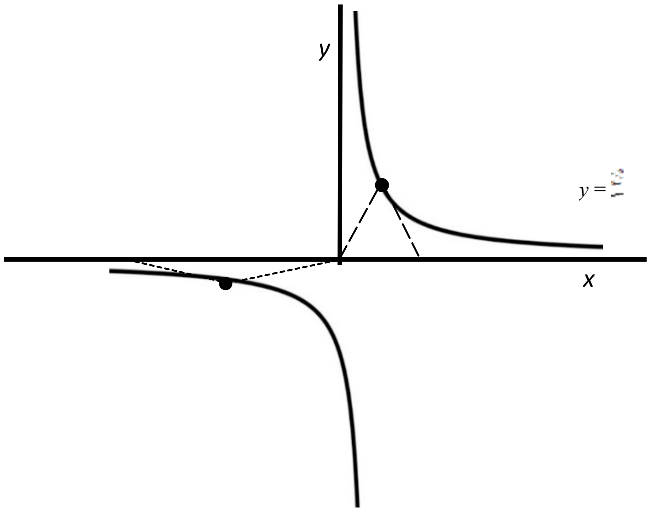
$\frac{dy}{dx} = \frac{6x^2}{y}$

$\frac{dy}{dx} = \frac{96}{16} = 6.$

1.3.2 Classroom Approach

This ratio of gradients can be approached in the classroom:

a) graphically, for example for $y = \frac{1}{x}$ the ratio of gradients is -1,



For $y = \frac{1}{x}$

the Ratio of Gradients = -1

b) by algebraic calculation of the ratio for specific cases,

c) algebraically to verify the general conclusion.

It is very motivating to see a conjecture suggested from a graph verified algebraically or vice versa.

2. AREAS UNDER CURVES

2.1 Modern Approach to Areas under Curves

First, integration is defined as the reverse of differentiation. Then various methods are used to show that an integral (or anti-derivative) of a function defines an area under that function.

We may argue as follows.

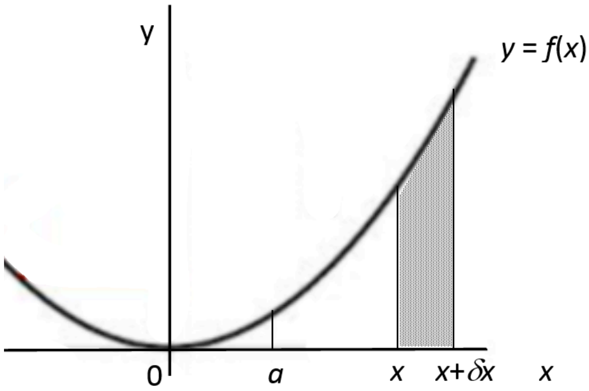
Let $A(x)$ = the area under $y = f(x)$

from $x = a$ to $x = x$, where $A(a) = 0$.

Then the area of the shaded strip on the

right = $A(x+\delta x)$ - $A(x) \approx \delta x\, f(x)$.

Therefore $\dfrac{A(x+\delta x)-A(x)}{\delta x} \approx f(x)$.



But $\lim\limits_{\delta x\to 0}\dfrac{A(x+\delta x)-A(x)}{\delta x} = A'(x)$, therefore in the limit $A'(x) = f(x)$.

And if $A'(x) = f(x)$ then $A = \int_a^x f(x)dx$.

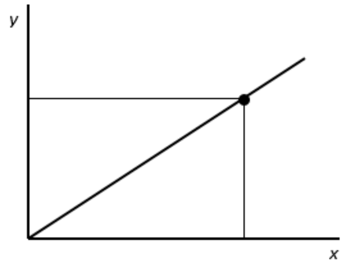
The full argument is not given here due to lack of space, but we see that we can find areas under $y = f(x)$ by integrating $f(x)$, substituting the limits and subtracting.

Next we look at an alternative approach that can be easily grasped by young children.

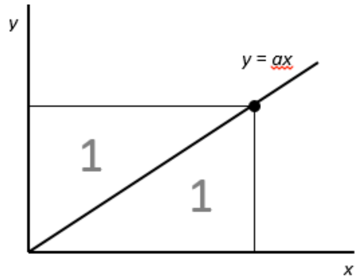
2.2 Vedic Approach to Areas under Curves

2.2.1 Ratios of Areas

Take a pair of axes and from the
origin draw a straight line as shown, with any gradient.
Select any point on the line and drop
perpendiculars onto the two axes, as also shown.



The two triangles generated will be equal in area, and this will apply for any straight line chosen: the ratio of areas will be 1:1.



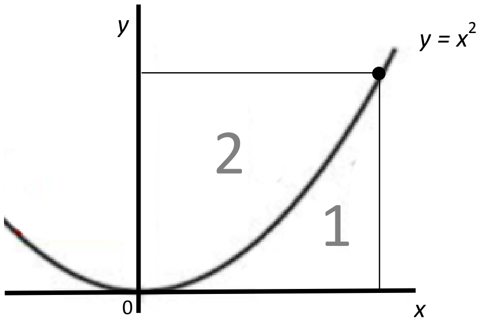
Now suppose we wish to draw a line, starting at the origin, but the requirement is that the ratio of areas similarly generated will be 2:1 rather than 1:1.

The line must clearly be a curve, rather than a straight line.

Also its gradient must be increasing if the area to the left of the line is to be double the area under it.

What will be the equation of the curve that has this special property?

Well, it turns out that the answer is the parabola given by $y = x^2$.

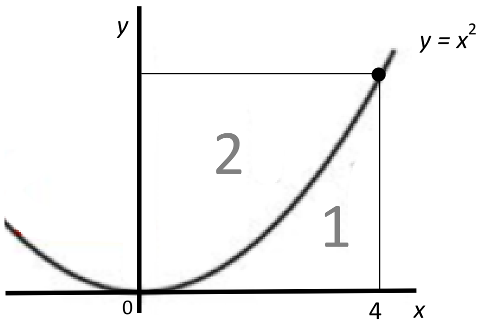


In fact any of the parabolas given by $y = ax^2$ will have this property.
We will demonstrate this result shortly.

This means that the area indicated by ‘1’ above is one third of the total area, and area ‘2’ will be two thirds of the total rectangle.

Example: Suppose we want the area under $y = x^2$ from $x = 0$ up to $x = 4$.

We can find the area of the rectangle and find one third of that.



The height of the rectangle is $4^2 = 16$, so the rectangle area is $4 \times 16 = 64$.

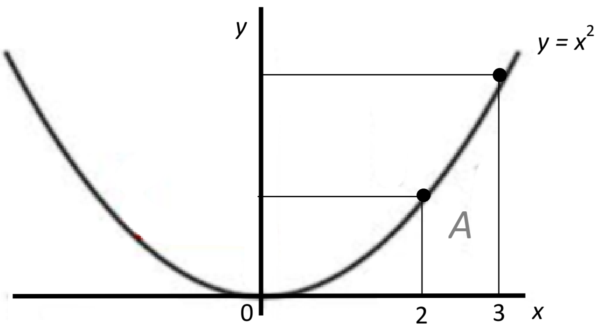
And the required area will therefore be $\frac{1}{3} \times 64 = \frac{64}{3}$.

In fact we can see that the area is one third of the cube of the x-coordinate.
This is just simple *Proportion*.

2.2.2 Area of a Strip

Similarly we can find the area of a strip A, like the area from $x = 2$ to $x = 3$ shown below.

Since we can find the area from the origin up to $x = 3$



and also the area up to $x = 2$,
we can find the difference of these,
which will be the required area.

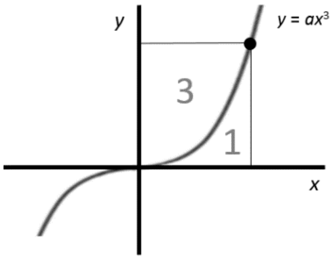
So we need $\frac{3^3}{3} - \frac{2^3}{3} = \frac{27}{3} - \frac{8}{3} = \frac{19}{3}$.

We get the area as $\frac{19}{3}$.
We simply subtract the cubes of the two x-values, and divide by 3.

2.2.3 Extending

We can also use symmetry in various ways to extend our range of techniques.

And we can extend to:



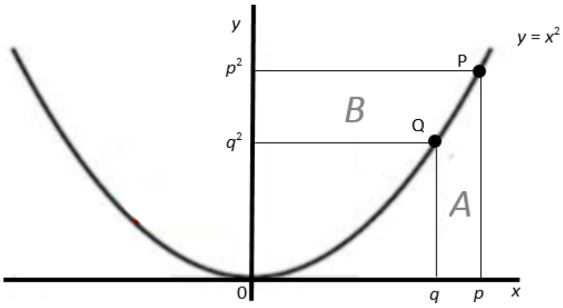
- $y = ax^2 + bx + c$ where we have a difference of cubes, difference of squares and difference of x-coordinates,
- $y = ax^3$ where the ratio of areas is 3:1,
- $y^2 = ax^3$ where the ratio of areas is 3:2.
- $y = \frac{a}{x^2}$, $x \neq 0$, where the ratio will be 2:1 (since areas are always positive).
- And in general for $y = ax^n$, $n\in$ § , $a\in$ ° , the ratio is n:1.

So for $y = ax^n$:

The ratio of these areas is *n*.

2.2.4 Explanation of Ratio of Areas = *n* for $y = x^2$

We can show that the ratio of areas for $y = x^2$ is 2:1 as follows.



That is, show that $\frac{B}{A} = 2$, where A
is the value of the area between the curve,
the x-axis and the lines $x = p$ and $x = q$.
And B is the value of the corresponding area,
as shown.

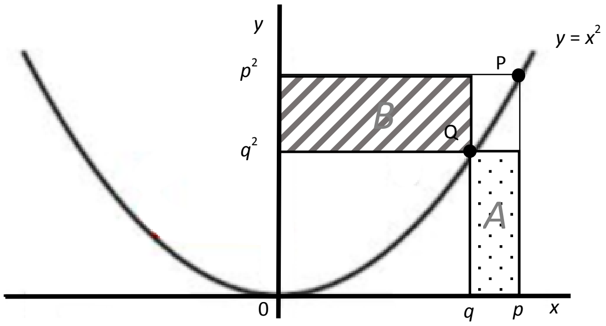
The explanation follows similar lines
to that for the gradient of a curve inasmuch

as we take two points $P(p, p^2)$, $Q(q, q^2)$, express the ratio of areas approximately and then let $q \to p$.

We approximate areas A and B with the rectangles shown shaded.
The ratio of these areas is

$$\frac{B}{A} = \frac{q(p^2-q^2)}{(p-q)q^2} = \frac{p+q}{q},$$

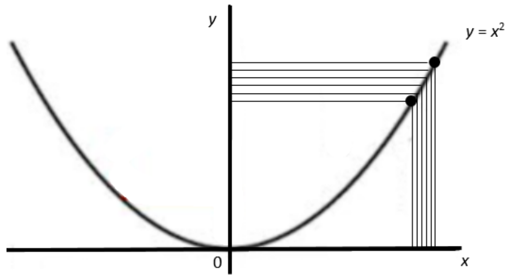
which in the limit goes to $\frac{2p}{p} = 2$.



This shows that as the horizontal and vertical strips get narrower and narrower the ratio of the areas approaches 2:1, and ultimately equals 2:1.

And since the ratio of areas is independent of x, it applies to all such strips.

Now any area to the left of $y = x^2$ and bounded by two horizontal lines and the y-axis (like area B) can be split into a large number of strips as shown.



And since the ratio of areas of two corresponding strips (one horizontal and one vertical) is ultimately 2:1, therefore the ratio of the whole area consisting of a collection of strips on the left, to the corresponding area below the curve will also be 2:1.

This shows that any area B in $y = x^2$ is double the corresponding area A.

2.2.5 GENERAL PROOF THAT Ratio of Areas = *n*

Given $y = ax^n$, where $n\in$ § , $a, x\in$ ° , take points on the curve $P(p, ap^n)$ and $Q(q, aq^n)$, $p, q\in$ ° .

Then $A = \int_q^p y\,dx = \int_q^p ax^n\,dx = \left[\frac{ax^{n+1}}{n+1}\right]_q^p$, $n \neq -1$*.

And $B = \int_{aq^n}^{ap^n} x\,dy = \int_{aq^n}^{ap^n} \left(\frac{y}{a}\right)^{\frac{1}{n}} dy = \left[\frac{an}{n+1}\left(\frac{y}{a}\right)^{\frac{1}{n}+1}\right]_{aq^n}^{ap^n}$, $n \neq -1$.

So $\frac{B}{A} = n\left(\frac{\left(p^n\right)^{\frac{n+1}{n}} - \left(q^n\right)^{\frac{n+1}{n}}}{p^{n+1} - q^{n+1}}\right) = n\left(\frac{p^{n+1} - q^{n+1}}{p^{n+1} - q^{n+1}}\right)$.

And $\lim_{q \to p} n\left(\frac{p^{n+1} - q^{n+1}}{p^{n+1} - q^{n+1}}\right) = n$.

* The case where n = -1 is straightforward.

3. SUMMARY and Concluding Remarks

We have seen that we can use limiting arguments to obtain the gradient of curves given by $y = ax^n$ and to obtain areas under them. Compared to traditional approaches this has certain advantages, notably that calculus can be taught to quite young students.

The secant-tangent method, and the memorable result that **Ratio of Gradients = Ratio of Areas = *n*** enable us to easily get gradients of a range of curves and areas under them.

By restricting the study to curves of the form $y = ax^n$ pupils can be exposed to the main principles of calculus at a young age, so that the understanding and development of this subject becomes a lot smoother, and a good foundation is prepared for the more advanced work which may follow.

This way pupils get the opportunity to develop and apply skills in algebra, limits etc. The necessary notation can be introduced as appropriate at the discretion of the teacher. 'Vedic' methods for more advanced work (like the product rule, chain rule, differential equations etc.) are also available so this approach can be developed further, and merged with the conventional approach at any point.

Reference

[1] Bharati Krsna Tirthaji Maharaja, " Vedic Mathematics", Motilal Banarasidas Publisher, Delhi, 1994.