Multiplication techniques: Ancient Indian methods vis-à-vis Tirthaji's methods in Vedic Maths

Arvind Prasad
Research Fellow, University of Queensland, Brisbane, Australia

Abstract

India holds a rich tradition in mathematics dating back to 2000 years and more. Sections on mathematics are contained in many ancient texts on astronomy - the *Shulva sutras* (~800 BCE) or in individually authored texts such as the *Aryabhatiya* (499 CE). In addition to this, simple arithmetic operations form part of Indian mathematics called *Paati-ganita,* contained in the Bakhshaali manuscripts (200 CE) and other texts.

Recently (1965), Bharathi Krishna Tirthaji wrote about 16 sutras and 13 upasutras in his seminal work 'Vedic Mathematics'. It is claimed that these sutras describe the natural way the mind works and thus cover all of mathematics. Consequently, the applications of the sutras range from simple arithmetic to the more advanced subjects of coordinate geometry and calculus.

It is of great interest to test if the proposed sutras of Tirthaji's Vedic Maths (VM) can be extended to the mathematics of the ancient times. In this paper the seven methods of *Paati-ganita* multiplication are compared with Tirthaji's methods of multiplication. The comparison reveals that the ancient method, *Tastha*, shows a one-to-one-match with *Urdhva Tiryagbhyamam* sutra method (Vertically and Crosswise). Furthermore, the application of some other sutras/upasutras such as *Purnapurnabhyam*, *Yavadunam* and *Anurupyena* can also be mapped to other ancient Indian techniques.

Introduction:

Following the exhaustive work of scholars such as Colebrook [1817], Smith [1923], Datta [1920s and 1930s], Gupta [1950s to 90s] and others, it is now well known that the subject of mathematics in India was held in high regard since ancient times and that the ancient mathematicians from the Indian civilization attained great heights in all areas of this subject.

Datta and Singh [1] develop a narrative of Indian mathematics starting from the ancient Vedic times dated several millennia prior to the Christian era to as late as 16th century of the common era, showing a constant growth in the knowledge on this subject. They conclude that the primary mathematical knowledge contained in the earliest Vedic texts provided the essential foundation for the subject to develop and prosper in the succeeding millennia. Bharathi Krishna Tirthaji (henceforth BKT), authored the seminal book 'Vedic Mathematics' [2], which provides a vastly different narrative to the mathematics contained in the Vedic texts. He provides 16 sutras (aphorisms) and 13 upa-sutras, which is claimed, follow the natural functioning of the mind thus covering the entire subject of mathematics. Therein it deviates substantially from Datta-Singh's (and other's) assertion. However, the original source – the *Vedic* texts, for both the authorship is the same.

Given the same source, with opposing views of the nature of mathematics, it is of interest to see if there is a correlation between the sutras of BKT to the arithmetic operations as used in pre-medieval India. Moreover, this differing view about the maths contained in the *Vedas* has led to a debate whether BKT's book has an appropriate name. This paper however only takes

an objective look at the possible commonalities and differences between the ancient Indian and BKT's system of mathematics.

To the best of the author's knowledge no such comparative work has been attempted before. Therefore as a first step, this comparison is performed by studying the multiplication techniques in the two systems. Following Datta-Singh's [1] reasoning that the pre-Christian era Vedic texts provided the source for subsequent medieval Indian mathematics, this paper defines 'ancient' as texts dating back to 1000 years or older.

History of Indian mathematics:

Interestingly, Datta also wrote an article titled 'Vedic Mathematics' [3]. His description of Vedic maths is based on literary works pre-dating the Christian era by as far back as a few millennia (3000 BC). The literary works comprise the *Vedas* and associated texts such as the *Samhitas*, *Brahamanas, Puranas*, *Vedaanga-Jyotish* and the *Shulva-sutras* (~800 B.C.). These texts have examples from astronomy, geometry, geometrical algebra and arithmetic. Work by Kak [4-6] and Achar [7-9] on the astronomy in the Vedic texts reaffirm Datta's conclusions. Starting from this ancient period, Datta and Singh [1] summarize the state of Indian mathematics up to the medieval times.

Mathematics in India has been called *ganita*. The first Vedic text, *Rig-Veda,* contains reference to the metric system, additive, subtractive and multiplicative principles for denoting numbers (both large and small) and the use of numbers to denote a thing. Some reference to the operation of division is also found in it. The *Samhitas*, *Puranas*, and *Siddhantas* repeat and add to the list of mathematics contained in the *Vedas*. The *Shulva-sutras* contain several works related to building altars of different geometrical shapes, the complexity of which led to solutions of a vast variety of mathematical problems: square-roots, cube-roots, indeterminate equations, etc. Several geometrical ideas such as area of trapezium, Pythagorean theorem (although preceding Pythagoras) etc. are also mentioned in them.

Indian mathematics has also been enriched by Buddhist and Jain canonical texts. The Buddhist literature classed *ganita* into *mudra* (finger), *ganana* (mental) and *sankhyaa* (higher arithmetic). The Jain work on *sankhyaa* is considered to be their highest achievement. Prior to the Christian era (~400 B.C.), following renaissance [10], higher arithmetic (*sankhyaa*) became *paati-ganita* (board maths). The name *paati-ganita* was derived from the use of chalk-slate to perform *ganita*, slate being a *paati*. *Sthaanaangasutra* mentions that the *ganita* subjects during this period included *Parikrama* ("Fundamental operations"), *Vyavahaara* ("Determinations"), *Rajju* ("Rope for geometry"), *Raashi* ("Rule of three"), *Kalaasvarna* ("Fractions"), *Yaavat-Taavat* ("Simple equations"), *Varga* ("Square and quadratic equations"), *Ghana* ("Cubes and cubic equations"), *Varga-Varga* ("Biquadratic equations") and *Vikalpa* ("Permutations and Combinations") [1].

The works that deal exclusively with *paati-ganita* are *Bhakshaali* manuscripts [200], *Trishatika [750]*, *Ganita-saar-samgraha [850]*, *Ganita-tilaka [1039]*, *Lilavati [1150]*, *Ganita-Kaumudi [1356]* and *Pati-saar [1658]*. The eight fundamental operations of *Parikrama* are: addition, subtraction, multiplication, division, square, square-root, cube, cube-root. Multiplication thus is a sub-section within *Parikrama*.

History of multiplication

Multiplication in India has been called *gunana* since ancient times and has been used in the Vedic literature. The *Shulva-sutras* refer to multiplication as *abhyasa* [1]. Other terms used were *hanana, vadha, kshaya* [notably, in the *Bhakshaali* manuscripts]. These all mean

'destroying' or 'killing' and apparently came into use following the place-value system and use of the concept of zero in multiplication problems wherein earlier numbers in the result would be rubbed out to be replaced by the latest number [1]. This erasing process gave it the synonym of 'destroying' and is akin to the VM sutra, Lopanasthapanabhyam, By elimination and retention.

Table 1. A list of Indian mathematicians from early times and the corresponding multiplication techniques which they have described. The English translation of the methods are in the footnote.

S.N.	Name	Year [C.E.]	Technique
1	Aryabhatta I	499	
2	Bhaskara I	525	
3	Brahmagupta	628	*Kapaata-Sandhi, Gomutrika, Ishta-gunana*
4	Shridhara	750	*Kapaata-Sandhi, Tastha*
5	Mahavira	850	*Kapaata-Sandhi, Tastha*
6	Aryabhatta II	950	*Kapaata-Sandhi*
7	Sripati	1039	*Tastha*
8	Bhaskara II	1150	*Kapaata-Sandhi, Sthaana-khanda, Ishta-gunana*
9	Narayana	1356	*Kapaata-Sandhi*
10	Ganesha	1545	*Kapaata-Sandhi, Tastha*

Kapaata-Sandhi – Door junction, Tastha – Standing, Ishta-gunana – Algebraic, Sthaana-khanda – Place separation, Gomutrika – Zig-zag

This paper only considers those ancient multiplication techniques which are part of the *paati-ganita* and described by the early mathematicians. These mathematicians are listed chronologically in Table I with the corresponding technique which they describe. This mathematical lineage is a strong indication of the development of mathematics as a subject of knowledge in ancient times in India.

Bharati Krishna Tirthaji's (BKT) Vedic Mathematics:

Tirthaji's book 'Vedic mathematics' was first published in 1965, half a decade after his death. In the Author's preface to his book titled "A descriptive prefatory note on the astounding wonders of ancient Indian Vedic mathematics" BKT says the following *"...after eight years of concentrated contemplation in forest-solitude, we {BKT} were at long last able to recover the long lost keys which alone could unlock the portals thereof {Vedic texts}."* He further states, *"And we were agreeably astonished and intensely gratified to find that exceedingly tough mathematical problems ... can be easily and readily solved with the help of these ultra-easy Vedic sutras..."*. He continues on the sutras, *"... there is no part of mathematics, pure or applied, which is beyond their {sutras'} jurisdiction; The Sutras are easy to understand, easy to apply and easy to remember"*.

These statements sum up Tirthaji's Vedic Mathematics. Briefly, it can be stated that following the careful study of the *Vedas*, BKT constructed 16 mathematical sutras (aphorisms pertaining to mathematics) and 13 upa-sutras (corollaries). According to him, this was achieved through contemplation of the Vedic text. A further claim is made that these aphorisms cover all branches of mathematics, pure and applied.

Ancient methods of multiplication:

Having introduced early mathematics in India, with an emphasis on the multiplication techniques, and the more recent Vedic Maths by Tirthaji, we now compare the multiplication methods from the two systems. The *paati-ganita* techniques (7 in total) are described first followed by BKT's sutra methods, each illustrated with the help of solved examples.

1] *Kapaata-sandhi* – Door Junction

There are two methods within this technique – Direct and Inverse. Examples for each are shown. The multiplicand is 53 and the multiplier is 47,.

1a) Direct method: In *paati-ganita*, the numbers were multiplied and the results put down in appropriate places replacing old numbers by erasing [1]. The numbers to be carried over were written separately or memorized. Here the process is closely replicated.

I	*II*	*III*	*IV*

```
   4 7            4 7          4 7
          ⇒               ⇒              ⇒   2  4  9  1
 5 3            5 4 1        5 4 1
```

I - The multiplier is placed above the multiplicand as shown. $47 * 3 = 141$ is carried out.

II - 3 is no longer required and 41 from 141 placed under 47 as shown. 1 from 141 (100s place) can be written at a different place or remembered.

III – The multiplier is moved to the left as shown and $47 * 5 = 235$ is carried out. Note from *II* and *III* that only the multiplicand digit under 4 is multiplied by the multiplier.

IV - 5 from 235 is added to the 4 from before; 4+5 = 9 is put down. 3 from 235 is added to the 1 from previous step, and 4 placed as shown. The final digit 2 from 235 comes adjacent to the left of 4. 2491 is the final answer.

1b) Inverse method: The modus operandi is similar but 47 placed differently than in the Direct method. $47 * 5 = 235$ is written as shown followed by moving 47 to the right. $47 * 3 = 141$ is carried out, 1 written under 7 and 14 from 141 added to 35 (from 235), giving a final answer of 2491.

I	*II*	*III*	*IV*

```
  4 7              4 7              4 7
          ⇒                  ⇒                ⇒   2  4  9  1
  5 3            2 3 5 3          2 3 5 3
```

The 'moving multiplier' methods suggest that the concept and the use of place-value system would have to be well-known and widely practised to be used in multiplication problems. Moreover, it shows multiplication being carried out from both Left-to-Right or Right-to-Left. Variations of the moving multiplier method is seen in VM methods as well [11].

2] Sridhar's *Kapaata-sandhi* [also *Gelosia* method]: Example with $53 * 47$ is shown below.

		5	3	
		2 / 0	1 / 2	4
		3 / 5	2 / 1	7
2	**4**	**9**	**1**	

The two digits of the multiplier and multiplicand are multiplied and results noted in boxes as shown. For instance, the shaded box shows the multiplication result for $7 * 3 = 21$, with 21 split across the cell with a diagonal. The resulting digits are added in the boxes along the diagonal direction which yields the final result, shown in bold.

3] *Tastha* method: Ganesha describes it thus in his commentary on *Lilavati* [1]:

That method in which the numbers stand in the same place, is called the Tastha gunana. After setting the multiplier under the multiplicand multiply unit by unit and note the result underneath. ... as in Vajrabhyasa multiply unit by ten and ten by unit, add together and set down the result in the line. Multiply unit by hundred,

hundred by unit and ten by ten, add together and set down the result as before; and so on with the rest of the digits. ... the line of results is the product.

'Tastha' means 'standing still' and thus the name bears the basic idea that the multiplier and the multiplicand stay in place as opposed to the *Kapaata-sandhi* method. Two things are readily apparent, 1] use of the place value system must have been well known since Ganesha does not explain this step in getting the final result and 2] the places for the digits had names.

$53 * 47$ is shown as a worked example for this method.

Adding digits in the corresponding places, i.e. (7+2) in the 10s place and (0+4) in the 100s, yields the final answer, 2491. Clearly, Left-to-Right or Right-to-Left operation is possible.

$$
\begin{array}{ccc}
5 & & 3 \\
4 & & 7 \\
\hline
(5*4) & [(5*7)+(3*4)] & (3*7)
\end{array}
$$

$$(20)\,|\,(47)\,|\,(21) \rightarrow 2491$$

4] *Sthaana-khanda*: $53 * 47$ is shown as an example.

$$
\begin{array}{cccc}
 & & 5 & 3 \\
 & & 4 & 7 \\
\hline
 & 3 & 7 & 1 \\
2 & 1 & 2 & \\
\hline
2 & 4 & 9 & 1
\end{array}
$$

The multiplicand and the multiplier are placed as shown. Starting from the right, $7 * 53$ and $4 * 53$ are carried out sequentially and the results noted as shown. Notice the shift in numbers 371 and 212 according to the place-value system. Hence the name *Sthaana-khanda*, translated as 'change in position'. This technique is also amenable to start from the Left or the Right hand side as long as the place-value system is used correctly.

5] *Gomutrika*: $53 * 47$ is shown below as an example.

$$
\begin{array}{ccc}
5 & 47 & 235 \\
3 & 47 & 141 \\
\hline
 & & 2491
\end{array}
$$

The multiplier and the multiplicand are written as shown. $5 * 47$ and $3 * 47$ are carried out and corresponding results noted as shown. The displacement of the product in rows I and II indicate the use of the place-value system. The products are then added keeping the place-value rules intact. Like previous examples, the product can be evaluated forwards or backwards. Moreover, the position of the multiplicand and the multiplier can be reversed. Clearly, working knowledge of the place-value system is necessary.

6] Parts multiplication method: Example with $12 * 53$ is shown in two different ways.

$12 * 53 = 3 * 4 * 53 = 636$

$$12 * 53 = (4*53) + (8*53) = 636$$

Here 12 is written in two different ways: 3*4 and (4+8). Breaking down 12 into smaller proportional parts (3,4) corresponds to the *Anurupyena* sutra in Vedic Maths.

7] *Ishta gunana*: Two versions of this technique are available and are shown below.

$12 * 53 = 53 * (12 + 8) - 53 * 8 = 636$ Here, $12 = (12 + 8 - 8)$.

$12 * 53 = 53 * (12 - 2) + 53 * 2 = 636$ Here, $12 = (12 - 2 + 2)$.

Note that 12 is deficient from 20, which requires 8 to be added to 12. These are examples of *Yavadunam* (deficiency) and *Purnapurnabhyam* (by completion or non-completion) upa-

sutras respectively in BKT's method. Likewise, $12 - 2 = 10$ is an example of *Purnapurnabhyam.*

Multiplication methods in the Vedic Maths of BKT:

In general, BKT's methods can be classified into two categories: general and special methods. There is one general method which can be applied to any multiplication problem. On the other hand, special methods, as the name indicates, are applicable for those questions only where certain features of the problem at hand can be made use of. Mathematical basis of these methods are provided elsewhere [12].

1] *Urdhva-Tiryak* – 'Vertically and Crosswise' (general method):

The pattern for this technique is shown with the help of a schematic for multiplication of two 2-digit numbers.

$$\begin{matrix} a & b \\ c & d \end{matrix} \rightarrow \begin{matrix} a \\ c \end{matrix} \Big| \begin{matrix} a & b \\ c & d \end{matrix} \Big| \begin{matrix} b \\ d \end{matrix}$$

$$\begin{matrix} 4 & 7 \\ 5 & 3 \end{matrix} \rightarrow \begin{matrix} 4 \\ 5 \end{matrix} \Big| \begin{matrix} 4 & 7 \\ 5 & 3 \end{matrix} \Big| \begin{matrix} 7 \\ 3 \end{matrix} \rightarrow 20\,|\,(35+12)\,|\,21 \rightarrow 20\,|\,47\,|\,21$$

The steps involved are, i} vertical multiplication (*ac*), ii} cross-wise multiplication and adding (*ad* + *bc*) and iii} vertical multiplication (*bd*). Following these steps for $53 * 47$, the final answer is 2491 after taking note of the place-value position of the result. Note the symmetricity in the technique i.e. steps {i} and {iii} can be interchanged and thus the method can be used Left-to-Right or Right-to-Left. Compare this with the ancient *Tastha* method.

2] Special methods – There are several other multiplication techniques in BKT's system where a method may be used when the appropriate opportunity arises. The following describes these special techniques categorized according to the sutra/upa-sutra used. Their specialities are illustrated with the help of examples. The English translation is also provided.

2a} *Vilokanam* – 'Mere Observation':

$$43 * 101 = 4343$$

The speciality of this method and the use of place-value system is easily noted.

2b} *Ekadhikena purvena* – 'By one more than the one before':

$$53 * 57 = 5 * (5+1)\,|\,3 * 7 = 30\,|\,21$$

$$75^2 = 7 * (7+1)\,|\,5 * 5 = 56\,|\,25$$

The sutra is seen in $5*(5+1)$ and $7*(7+1)$, 5 and 7 being the digit before the last digit respectively, in each case. The special case is the summation of 1s digits adding to 10: $3+7=10$ and $5+5=10$.

2c} *Anurupyena* – 'Proportionately':

$$6 * 23 = 2 * 3 * 23 = 138$$

Here, $6 = 2 * 3$. Thus this method is only appropriate when at least one number is non-prime.

2d} *Vyashthi-Samashthi* – 'By average {Specific and general}';

$$53 * 47 = 50^2 - 3^2 = 2491$$

The average of {53,47} is 50 and the difference between the average and both the numbers is 3. This method is helpful when the average is a multiple of 10 (e.g. 50). In any other case, the ease of the method is nullified, and hence its categorization as a special method.

2e} *Nikhilam Navatashcharmam Dashataha* – 'All from 9 and the last from 10': Also called *Nikhilam* in short and is used in subtracting numbers from the nearest base.

$88 * 92$ is shown as an example. 100 is used as a base, being close to both 88 and 92.

$$
\begin{array}{cc}
88 & -12 \\
92 & -08 \\
\hline
[88+(-08)] & [-12*-08]
\end{array}
$$

$$80 \mid 96 = 8096$$

The two numbers to be multiplied are written as shown with their respective deficiencies from the base, (-12 and -08), written in the adjacent column and obtained using the *Nikhilam* sutra. The answer is obtained in two steps/sides. The LHS in the answer is obtained by performing cross-subtraction [88+(-08)]. LHS can also be obtained by alternative operations: cross-subtraction [92+(-12)] or vertical addition. The RHS of the answer is obtained by vertically multiplying the deficiencies [(-12)*(-08)]. The elements of *Urdhva-Tiryak* sutra are clearly seen. Using the deficiencies of a number comes under the *Yavadunam* sutra.

This method offers a vast range of applications. For e.g., the numbers can be close to a multiple of a metric base (base 90 for 88*92). One such extensions is a problem of the type: $184 * 97$. Here the upa-sutra *Anurupyena* may be used to write 184 = 92*2 and use 100 (or 90) as base to multiply $92 * 97$. Multiplication by 2 in the final step yields the final result. Here, a combination of sutras are used: *Nikhilam* to calculate the deficiencies, *Yavadunam* to use the deficiencies, and *Anurupyena* to divide the problem into a smaller part.

Comparison of the ancient method and BKT's Vedic Maths techniques:

Both the systems use several methods for multiplying. Here, the specific similarities and differences in the techniques in the two systems are discussed.

Similarities between the two systems:
- Both systems provide the choice of left-to-right/right-to-left calculations (ancient system: *Kapaat-sandhi*, *Tastha*, *Sthaana-khanda* and *Gomutrika;* BKT: *Urdhva-Tiryak* sutra). This choice of direction of calculation in either system shows the continuity of the knowledge and practice of the place-value system.
- The significant finding of this study is that the *Tastha* method of the earlier times has a one-to-one match with the general Vedic Maths method (*Urdhva-Tiryak* sutra).
- Parts multiplication method of the ancient times was found to be similar to the BKT method of using the *Anurupyena* upa-sutra. Moreover, several modified versions of the moving multiplier method of *Kapaata-sandhi* is also part of the VM curriculum.
- The ancient *Ishta gunana* technique can be classified as 'specific' method of the modern Vedic Maths. Thus the ancient system can also be thought of as consisting specific and general methods.

Dissimilarities between the two systems:

The use of *Nikhilam*, *Vyashthi-Samashthi* and *Ekadhikena Purvena* sutras in BKT's Vedic system are not found in the ancient methods. In particular, the *Nikhilam* method has

applications in a vast range of multiplication problems which is not present in the *paati-ganita* system. With regards to these missing techniques, perhaps the early mathematicians considered such techniques to be too simple to comment upon. For instance, Aryabhata I and Brahmagupta only refer to the steps for finding square and cube roots and cube-roots respectively, neglecting all other arithmetic operations [1]. Thus what may be perceived as 'missing' in the ancient system may be a matter of simple operations having been omitted.

BKT's method seems to provide a basis for a given technique through the sutras. However, the descriptions of the arithmetic operations described in the ancient texts are only statements of the steps for that technique with no insight available on how the methods were developed. This apparent difference cannot be commented upon any further since the ancient methods may indeed have the sutras as their basis.

Concluding remarks:

Both Datta and Tirthaji point to the Vedic origins to Indian mathematics, but offer different points of view. As a starting point to the comparison between the mathematical techniques contained in the two systems, the current paper looked at the multiplication techniques in ancient Indian mathematics and the recent Vedic Maths methods as described by BKT.

Despite the differing points of view, several similarities were found in the techniques used in the two systems. These include a host of multiplication methods in both systems, Right-to-left and Left-to-Right calculations which require a good working knowledge of the place-value system, one-on-one match between the *Tastha* (ancient system) and *Urdhva Tiryak* method of BKT. Furthermore, it appears that the specific and general methods of multiplication exist in both systems, though not specifically stated in the ancient system. This might be perceived as a difference.

The significant commonalities between the two systems points to a continuity in the Indian tradition of mathematics and the techniques, at least with multiplication methods. Therefore, the natural extension of this work is to compare the fundamental operations of *Parikrama* between the two systems.

References:
[1] Datta B, Singh AN. History of Hindu Mathematics: Bharatiya Kala Prakashan, 2004.
[2] Tirthaji BK. Vedic Mathematics: Motilal Banarsidas, 2001.
[3] Datta B. Vedic Mathematics. In: Ray P, Sen SN, editors. The Cultural Heritage of India: Science and Technology. Calcutta: The Ramakrishna Mission Institute of Culture, 1984.
[4] Kak SC. *Indian Journal of History of Science* 1993;28:15.
[5] Kak SC. *Indian Journal of History of Science* 1998;33:175.
[6] Kak SC. *Indian Journal of History of Science* 1998;33:93.
[7] Achar BNN. *Indian Journal of History of Science* 1998;33:101.
[8] Achar BNN. *Indian Journal of History of Science* 2000;35:1.
[9] Achar BNN. *Indian Journal of History of Science* 2000;35:173.
[10] Datta B. *Indian Historical Quarterly* 1929;V:479.
[11] Williams K. Discover Vedic Mathemtics: A concise introduction to the Vedic system: Inspiration books, 2009.
[12] Babajee DKR, Williams KR. Vedic Mathematics Proofs: Validation of the system of Vedic Mathematics, 2013.