EXPOSITION OF THE CHAKRAWAL METHOD

(A Gem of Bhaskara's Algebra)

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Abstract

The Chakrawal method (an ancient Indian Cyclic method) is one of the greatest contributions of Bhaskara II to the field of Algebra. Bhaskara II (1150 AD) deals with the topic extensively in his treatise called Bija Ganitam. In it he determines positive integer solutions of indeterminate quadratic equations in the form $Nx^2 + 1 = y^2$. The Chakrawal method is an improvement upon the solution of this equation by Brahmagupta (598 AD) in the Varga Prakrati. Narayana Pandit (1356) solved a similar equation in his Ganita-Kaumudi.

In 1657 the famous French mathematician, Pierre de Fermat, proposed a solution to the equation $61x^2 + 1 = y^2$ to his countryman and fellow mathematician Bernard Frénicle de Bessy. The Swiss mathematician Leonhard Euler solved an equation in the form $x^2 - ny^2 = 1$ in 1732. He referred to it as Pell's equation, because he mistakenly thought the solution was due to an English mathematician called John Pell. However, the very same equation was solved by Bhaskara II about five hundred years earlier, and therefore may rightly be re-named the Bramagupta-Bhaskara equation.

Key Words

Pell's equation, Brahmagupta, Varga Prakrati, Bhaskara II, Bija Ganitam, Chakrawal method

Introduction

Pell's equation is any Diophantine equation of the form $x^2 - ny^2 = 1$. It was first studied extensively in India. In his Brahma Sputa Siddhanta about a thousand years before Pell's time, Brahmagupta developed a Chakrawal (cyclic) method to help solve such an indeterminate quadratic equation. Although Brahmagupta came close to solving the equation, his solution was improved upon by Bhaskara II in his treatise Bija Ganitam .

Many links to the past lie buried in the sands of time. It is unfortunate that the important contributions of ancient Indian mathematicians, such as Brahmagupta and Bhaskara II, are greatly overlooked or often even forgotten. The objective of this paper is to create awareness among students and teachers of the work of these great mathematicians.

Indeterminate Equations

When the number of unknown quantities is greater than the number of independent equations, there is an unlimited number of solutions. Such equations are said to be indeterminate. The second order equation $Nx^2 + 1 = y^2$ falls into this category.

Brahmagupta

Brahmagupta is generally recognized as an Indian astronomer and mathematician par excellence. Born in 598 AD, Brahmagupta was a resident of Bhinamala, which is near the northern border of Gujarat in south Rajasthan. His well-known work and masterpiece "Brahma Sputa Siddhanta" was written in 628 AD. This treatise comprises 1008 verses in 24 chapters. The eighteenth chapter - consisting of 102 verses - is called Kuttaka, which literally means "pulverize". This chapter discusses the solution of indeterminate quadratic equations (called Varga Prakrati) of the type $Nx^2 + 1 = y^2$. The word Varga (meaning square) refers to the variable x^2, while the word Prakrati refers to the coefficient N of the x^2.

Bhaskara II

The celebrated twelfth century mathematician Bhaskaracharya is generally referred to as Bhaskara II, to distinguish him from his seventh century namesake Bhaskara I. He was born in 1114 A.D. and lived in Vijjada Vida in Karnataka. Bhaskara II greatly admired and was influenced by Brahmagupta's work. He conferred on him the coveted title "Ganaka Chakra Chudamani", which means "The Jewel in the Galaxy of Mathematicians".

Bhaskara's celebrated work, Siddhanta Siromani, consists of four parts called Lilawati, Bija Ganitam, Graha Ganitam and Goladhyaya. Bija Ganitam, a treatise on advanced algebra, contains an exposition of the Indian cyclic method called Chakrawal. Bhaskara II used this method to solve the equation $Nx^2 + 1 = y^2$.

The Brahmagupta – Bhaskara Equation

An indeterminate quadratic equation of the form

$$Nx^2 \pm k = y^2$$

has integer values of N and k, where $N > 0$. N may not be a square number.

In attempting to solve this equation, Brahmagupta recognized that a positive integral solution of $Nx^2 \pm k = y^2$ can always be found if $k = \pm 1, \pm 2$ *or* ± 4. For the ensuing discussion, some important lemmas of Brahmagupta need to be considered.

Brahmagupta's Lemmas

Lemma 1: If $x = \alpha$ and $y = \beta$ is a solution of the equation $Nx^2 + k = y^2$ and if $x = \alpha_1$ and $y = \beta_1$ is a solution of the equation $Nx^2 + k_1 = y^2$, then $x = \alpha\beta_1 \pm \alpha_1\beta$ and $y = \beta\beta_1 \pm N\alpha\alpha_1$ is a solution of the equation $Nx^2 \pm kk_1 = y^2$.

Lemma 2: If (α, β) is a solution of the equation $Nx^2 + k = y^2$, then $(x, y) = (2\alpha\beta, N\alpha^2 + \beta^2)$ is a solution of the equation $Nx^2 \pm k^2 = y^2$

Lemma 3: If $x = \alpha$ and $y = \beta$ is a solution of the equation $Nx^2 + k^2 = y^2$, then $x = \dfrac{\alpha}{k}$ and $y = \dfrac{\beta}{k}$ is a solution of the equation $Nx^2 + 1 = y^2$.

The above results are technically called "Bhavana". They are further categorized as Samasa Bhavana (additive composition) and Antara Bhavana (subtractive composition). When the Bhavana is made with two equal sets of roots and interpolations, it is called Tulya Bhavana (composition of equals). Similarly, the composition with unequal sets of roots is called Atulya Bhavana (composition of unequals).

From Brahmagupta's Lemma 1 it is clear that when two solutions of Varga Prakrati $Nx^2 + 1 = y^2$ are known, any number of other solutions can be obtained. If (a_1, b_1) and (a_2, b_2) are two solutions of $Nx^2 + 1 = y^2$, then $x = a_1b_2 \pm b_1a_2$ and $y = b_1b_2 \pm Na_1a_2$ are two other solutions.

Combining this solution with previous ones, we get other solutions. Furthermore, Brahmagupta's Lemma 2 states that if (a, b) is a solution of the given equation, then another solution is $(2ab, b^2 + Na^2)$.

It can therefore be seen that, to obtain a set of solutions of a given equation of the type $Nx^2 + 1 = y^2$, it is necessary to obtain only one solution to start with. After that, infinitely many solutions can be obtained by repeated application of Brahmagupta's Bhavana.

Eample: What is that square which multiplied by 8 becomes, together with unity, a square?

The following equation needs to be solved:

$$8x^2 + 1 = y^2$$

By the Vedic sutra *Vilokanam* (observation), one solution is $(x, y) = (1, 3)$, as

$$8(1)^2 + 1 = 3^2$$

By lemma 2, if (α, β) is a solution of the equation $Nx^2 + k = y^2$, then $(x_1, y_1) = (2\alpha\beta, N\alpha^2 + \beta^2)$ is also a solution. With $\alpha = 1$ and $\beta = 3$

$$(x_1, y_1) = [2(1)(3), 8(1)^2 + (3)^2] = (6, 17)$$

Then by Lemma 1, we can obtain a new solution by the composition (Bhavana) of the two solutions.

If $a_1 = 1$ and $b_1 = 3$

and $a_2 = 6$ and $b_2 = 17$

Then a new solution

$(x_2, y_2) = (a_1b_2 + b_1a_2, Na_1a_2 + b_1b_2)$

$= (35, 99)$

In this manner, we can find an infinite number of solutions $(x, y) = (1, 3), (6, 17), (35, 99), \dots$

In Brahmagupta's method, the only drawback is that there is a need for a trial solution to start with. This problem is ably addressed by the remarkable Chakrawal method of Bhaskara-II.

The Chakrawal Method of Bhaskara II

To solve $Nx^2 + 1 = y^2$, we can find solutions, a and b, such that

$$Na^2 + k = b^2 \qquad (1)$$

We also have

$$N(1)^2 + (m^2 - N) = m^2 \qquad (2)$$

Let the respective solutions of the two equations be (a_1, b_1) and (a_2, b_2). Let $a_1 = a$ and $b_1 = b$. Let $a_2 = 1$ and $b_2 = m$.

Using Brahmagupta's Lemma

If $x = a$ and $y = b$ is a solution of the equation $Nx^2 + k = y^2$ and if $x = 1$ and $y = m$ is a solution of the equation $Nx^2 + (m^2 - N) = y^2$, then $x = am + b$ and $y = Na + bm$ is a solution of the equation $Nx^2 + k(m^2 - N) = y^2$.

We can thus also write

$$N(am + b)^2 + k(m^2 - N) = (Na + bm)^2$$

We have used Samasa Bhavana (additive composition).

On dividing both sides of the latter equation by k^2 we obtain

$$N\left(\frac{am+b}{k}\right)^2 + \frac{m^2-N}{k} = \left(\frac{bm+Na}{k}\right)^2 \qquad (3)$$

The Kuttaka ("pulverize") method is now applied:

Choose m such that $am + b$ is divisible by k and $\frac{m^2-N}{k}$ is the smallest possible integer.

Let $\frac{am+b}{k} = a_1$, $\frac{m^2-N}{k} = k_1$, $\frac{bm+Na}{k} = b_1$

Equation (3) can then be written as

$Na_1^2 + k_1 = b_1^2$ where a_1, b_1, k_1 are integers.

The full process described above can be repeated over and over again.

This is the cyclic Chakrawal method of Bhaskara-II.

Bhaskara's Theorem 1: When a_1 is an integer, then b_1 and k_1 are also integers.

Bhaskara's Theorem 2: After a finite number of repetitions, two integers α and β can obtained such that $N\alpha^2 + \lambda = \beta^2$ where $\lambda = \pm 1$ or ± 2 or ± 4.

Thus, starting with $Na^2 + k = b^2$, where k is any convenient integer, we can arrive at a solution (α, β) of the equation where λ takes the value 1, 2 or 4 with either a positive or a negative sign.

Once this solution is obtained, Brahmagupta's usual method will lead to an integral solution of the given equation $Nx^2 + 1 = y^2$

Examples Worked Out by Bhaskara II - The Bija Ganitam

Example 1: What is that number whose square multiplied by 61 and then added to 1, yields a perfect square?

We must solve $61x^2 + 1 = y^2$

Assume a parallel equation of the form $61a^2 + k = b^2$

When $a = 1$, $k = 3$, $b = 8$ and $N = 61$ we obtain, $61(1)^2 + 3 = 8^2$

Thus, $\dfrac{am+b}{k} = \dfrac{m+8}{3}$ and $\dfrac{m^2-n}{k} = \dfrac{m^2-61}{3}$

Substituting the above expressions into equation (3) of Bhaskara-II

$$N\left(\frac{am+b}{k}\right)^2 + \frac{m^2-N}{k} = \left(\frac{bm+Na}{k}\right)^2 \tag{3}$$

we obtain

$$(61)\left(\frac{m+8}{3}\right)^2 + \frac{m^2-61}{3} = \left(\frac{61+8m}{3}\right)^2$$

Now find m so that $\dfrac{m+8}{3}$ is an integer (i.e. divisible by 3) and $\dfrac{m^2-61}{3}$ is a small integer.

$\dfrac{m+8}{3}$ is an integer for m = 1, 4, 7, 10…

and $\dfrac{m^2-61}{3}$ yields a small integer (i.e. -4) for m = 7

Choosing $m = 7$, the equation

$$(61)\left(\frac{m+8}{3}\right)^2 + \left(\frac{m^2-61}{3}\right) = \left(\frac{61+8m}{3}\right)^2$$

reduces to

$$61.\ 5^2 - 4 = 39^2$$

This equation is in the form $Nx^2 + \lambda = y^2$ where $\lambda = -4$

On dividing by 4, we obtain

$$(61)\left(\frac{5}{2}\right)^2 - 1 = \left(\frac{39}{2}\right)^2$$

Thus we obtain a solution $(x, y) = (\frac{5}{2}, \frac{39}{2})$ to the equation $61x^2 + 1 = y^2$

Furthermore, by using Brahmagupta's Lemma 2,

If (α, β) is a solution of the equation $Nx^2 + k = y^2$, then $(x, y) = (2\alpha\beta, N\alpha^2 + \beta^2)$ is a solution of the equation $Nx^2 \pm k^2 = y^2$

With $\alpha = \frac{5}{2}$, $\beta = \frac{39}{2}$, $k = -1$ and $N = 61$

We obtain $2\alpha\beta = \frac{195}{2}$ and $N\alpha^2 + \beta^2 = \frac{1523}{2}$

Substitution of these values into the equation

$$61x^2 + 1 = y^2$$

yields $$61(2\alpha\beta)^2 + k^2 = (N\alpha^2 + \beta^2)^2$$

thus $$61\left(\frac{195}{2}\right)^2 + (-1)^2 = \left(\frac{1523}{2}\right)^2$$

or $$61\left(\frac{195}{2}\right)^2 + 1 = \left(\frac{1523}{2}\right)^2$$

Next, by performing the Samasa Bhavana between

$$\frac{195}{2}, \frac{1523}{2} \text{ and } +1 \qquad \text{and} \qquad \frac{5}{2}, \frac{39}{2}, -1$$

we obtain $\alpha = 3805$, $\beta = 29718$ and $\lambda = -1$

(Samasa between a_1, b_1, k_1 and a_2, b_2 k_2 gives $\alpha = a_1b_2 + b_1a_2$, $\beta = Na_1a_2 + b_1b_2$ and $\lambda = k_1k_2$.)

Lastly, applying Lemma 2 on the set $\alpha = 3805$, $\beta = 29718$ and $\lambda = -1$, we obtain a further solution

$x = 226153980$ and $y = 1766319049$ $\qquad$ ($x = 2\alpha\beta$ and $y = N\alpha^2 + \beta^2$, with $k^2 = (-1)^2 = 1$)

This is the smallest integral solution of the equation $61x^2 + 1 = y^2$ (having 9 and 10 digits respectively).

Infinitely many solutions can be obtained by repeated application of Samasa Bhavana.

Summary

To solve $Nx^2 + 1 = y^2$

Find a and b for a suitable k to form the auxiliary equation $Na^2 + k = b^2$ having the triple (a, b, k) and set $N(1)^2 + (m^2 - N) = m^2$ having the triple $(1, m, m^2\text{-}N)$.

By composition of triples, we have a new (Samasa) triple $[am + b, Na + bm, k(m^2\text{-}N)]$.

We then obtain

$$N\left(\frac{am+b}{k}\right)^2 + \frac{m^2 - N}{k} = \left(\frac{bm+Na}{k}\right)^2$$

By the Kuttaka method, we choose 'm' so that $\frac{am+b}{k}$ is an integer and $|m^2 - N|$ is a minimal integer.

Writing $a_1 = \frac{am+b}{|k|}$, $b_1 = \frac{bm+Na}{|k|}$, $k_1 = \frac{m^2 - N}{k}$

we then have $Na_1^2 + k_1 = b_1^2$

Using a_1, b_1 and k_1 instead of a, b and k, we now repeat the above process, obtaining another set of integers a_2, b_2 and k_2 such that $Na_2^2 + k_2 = b_2^2$.

The process is repeated until a triple with k = 1, 2 or 4 is found (k can have a negative value also). Brahmagupta's approach thus gives a final solution.

Example 2: What is that number whose square multiplied by 67 and then added to unity yields a perfect square?

To solve $67x^2 + 1 = y^2$, for the first iteration assume an auxiliary equation $67a^2 + k = b^2$.

By inspection we find that the triple (*1, 8, -3*) satisfies this equation, as substitution of $a = 1$, $b = 8$, $k = -3$ and $N = 67$ yields

$67(1)^2 - 3 = 8^2$

If $m = 7$, then $\frac{m+8}{-3}$ is an integer. Also, $\left|\frac{7^2 - 67}{-3}\right|$ is a minimum integer.

This gives $a_1 = 5$, $b_1 = 41$, $k_1 = 6$ and $N = 67$

Thus $\quad 67.\ 5^2 + 6 = 41^2$

For the second iteration $(a, b, k) = (5, 41, 6)$

For $m = 5$, $\frac{41+5m}{6}$ is an integer and $\left|\frac{m^2-6}{6}\right|$ is a minimal integer.

This leads to a new solution with $a_2 = 11$, $b_2 = 90$ and $k_2 = -7$, i.e.

$67.\ 11^2 - 7 = 90^2$

For the third iteration $(a, b, k) = (11, 90, -7)$

For $m = 9$, $\frac{90+11m}{-7}$ is an integer and $\left|\frac{m^2-67}{-7}\right|$ is minimal.

This leads to a new solution with $a_3 = 27$, $b_3 = 221$ and $k_3 = -2$, i.e.

$67.\ 27^2 - 2 = 221^2$

We have thus reached one of the prescribed values of k, consisting of the triple $(27, 221, -2)$.

With this triple, we use Lemma 2 to obtain the integral solution $(x, y) = (11934, 97684)$ to the equation $67x^2 + 4 = y^2$.

(Note: $k_3^2 = (-2)^2 = 4$).

Dividing $(x, y) = (11934, 97684)$ by 2, we then obtain the smallest integral solution $(x, y) = (5967, 48842)$ to the equation $67x^2 + 1 = y^2$.

Historical Remarks

One of the most important contributions of ancient Hindus to mathematics, is in perfecting the method of finding integer solutions to indeterminate second-degree equations of the type $Nx^2 + 1 = y^2$. Bhaskara II's Chakrawal (cyclic) method was an improvement on the prior work of Brahmagupta. In his Bija Ganitam, Bhaskara found solutions to the equations $67x^2 + 1 = y^2$ and $61x^2 + 1 = y^2$. Narayana Pandit (1356) in his Ganita-Kaumudi solved the equations $97x^2 + 1 = y^2$ and $103x^2 + 1 = y^2$. These indeterminate equations are also called Diophantine equations, after the Greek mathematician Diophantus of Alexandria at about the third century AD. A part of the work Arithmetica of Diophantus was studied by Fermat. The famous French mathematician, Fermat, in a letter of February 1657 AD, challenged Frenicle and other European mathematicians to find the solution to the equation $61x^2 + 1 = y^2$. None of them succeeded in solving the equation in integers. It was only in 1732 that Euler found a complete solution. The complete theory underlying the solution was expounded by Lagrange in 1767.

About five hundred years earlier, the very same equation was completely solved by Bhaskara II. There is no evidence to prove that Fermat knew about Brahmagupta-Bhaskara work. Colebrook's English translation of Bhaskara's Bija Ganitam was published only in 1817.

Conclusion

This remarkable achievement of Brahmagupta and Bhaskara II to obtain integral solutions of indeterminate equation by Chakrawal method is indeed praiseworthy. Under these circumstances it may be appropriate to designate the equation of the form $Nx^2 + 1 = y^2$ as the Brahmagupta-Bhaskara equation. It is hoped that this paper lead to a proper appreciation of the great contribution of ancient Indian Mathematicians

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