How the Binomial Theorem Underlies the Working of the Anurupyena and Yavadunam Sutras in the Calculation of Successive Powers of a Number; Application of power Triangles and Calculus

Marianne Fletcher and James Glover

Abstract

"The binomial theorem is thus capable of practical application and – in its more comprehensive Vedic form – has thus been utilized, to splendid purpose, in the Vedic Sutras. And a huge lot of Calculus work both differential and integral has been and can be facilitated thereby…"

"Vedic Mathematics" Tirthaji

The working of the Anurupyena and Yavadunam sutras, in the calculation of squares, cubes and higher powers of numbers, is demonstrated to be an application of the binomial theorem.

Pascal's arithmetic triangle (with pinnacle "11") is conventionally used to help calculate binomial coefficients. The powers of 11 are also directly generated from the successive rows in the arithmetic triangle. The question arose whether a pascal-type triangle, with a number at its pinnacle containing digits other than "11", might be used to help calculate successive powers of the particular number.

It was found that such a "power triangle" - with the number "*ab*" at its pinnacle - does not directly generate the powers of "*ab*". However, the numbers in its successive rows can, indeed, with the help of the Anurupyena sutra, be employed to calculate any power of "*ab*".

Athough this method is more laborious than direct application of Tirthaji's sutras, there are some special cases where the n-th row in a power triangle can be directly used to find the *n*-th power of a number.

It is of interest to note that the sum of any power triangle and its "complement" (after applying the Nikhilam sutra to the number at the top) always yields Pascal's original "11" arithmetic triangle again.

The discussion ends with a brief look at how successive differentiations and integrations of the variables x and y yield the terms in the binomial expansion of $(x+y)^n$. Employing such a method, particularly for fractional and negative indices, can be a useful alternative approach to finding the terms in a binomial expansion. It is easy to remember and easy to apply.

Background

The expansion of a binomial $(a+b)^n$ into $n+1$ terms is given by:

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

where $\binom{n}{k}$ or ${}^n_rC$ is the binomial coefficient which can be calculated using the formula

$$\binom{n}{k} = \frac{n!}{(n-r)!r!}.$$

For example,

$$(a+b)^4 = \binom{4}{0}a^4b^0 + \binom{4}{1}a^3b^1 + \binom{4}{2}a^2b^2 + \binom{4}{3}a^1b^3 + \binom{4}{4}a^0b^4$$
$$= 1a^4 + 4a^3b^1 + 6a^2b^2 + 4a^1b^3 + 1b^4$$

The binomial coefficients can be generated from Pascal's arithmetic triangle:

Row 0:		$\binom{0}{0}$
Row 1:	1 1	$\binom{1}{0}$ $\binom{1}{1}$
Row 2:	1 2 1	$\binom{2}{0}$ $\binom{2}{1}$ $\binom{2}{2}$
Row 3:	1 3 3 1	$\binom{3}{0}$ $\binom{3}{1}$ $\binom{3}{2}$ $\binom{3}{3}$
Row 4:	1 4 6 4 1	$\binom{4}{0}$ $\binom{4}{1}$ $\binom{4}{2}$ $\binom{4}{3}$ $\binom{4}{4}$
Row 5:	1 5 10 10 5 1	$\binom{5}{0}$ $\binom{5}{1}$ $\binom{5}{2}$ $\binom{5}{3}$ $\binom{5}{4}$ $\binom{5}{5}$
Row 6:	1 6 15 20 15 6 1	$\binom{6}{0}$ $\binom{6}{1}$ $\binom{6}{2}$ $\binom{6}{3}$ $\binom{6}{4}$ $\binom{6}{5}$ $\binom{6}{6}$

One common use of the binomial expansion is to calculate powers of, say, 23. For example, 23^4,

$$(20+3)^4 = \binom{4}{0}(20)^4(3)^0 + \binom{4}{1}(20)^3(3)^1 + \binom{4}{2}(20)^2(3)^2 + \binom{4}{3}(20)^1(3)^3 + \binom{4}{4}(20)^0(3)^4$$
$$= 1(2^4)(10^4) + 4(2^3)(3^1)(10^3) + 6(2^2)(3^2)(10^2) + 4(2^1)(3^3)(10^1) + 1(2^0)(3^4)$$
$$= 160000 + 96000 + 21600 + 2160 + 81$$
$$= 279841$$

Anurupyena and the Binomial Theorem

In his chapter on Elementary Squaring and Cubing, Tirthaji in effect makes use of the binomial expansion when he applies Anurupyena (Proportionally) to calculate the values of the 3rd and 4th powers of a two digit number like 23

For $(23)^3$

$$\begin{array}{cccc} (2)^3 & & & (3)^3 \\ 8 & 12 & 18 & 27 \end{array}$$
$$\xrightarrow{\times\frac{3}{2}} \quad \xrightarrow{\times\frac{3}{2}} \quad \xrightarrow{\times\frac{3}{2}}$$

where $12 = (2)^2(3)^1$ and $18 = (2)^1(3)^2$

Tirthaji then proceeds to add 2×12 and 2×18 to the two middle terms, i.e.

$$\begin{array}{cccc} 8 & 12 & 18 & 27 \\ & 24 & 36 & \\ \hline 8 & 36 & 54 & 27 \end{array} = 12167$$

The final answer 12167 is then obtained when the "tens digits" are taken to the left and added.

It is clear that the binomial expansion is at work here, as:

$$\begin{array}{cccc} 8 & 12 & 18 & 27 \\ & 12 & 18 & \\ & 12 & 18 & \\ \hline 1(8) & 3(12) & 3(18) & 1(27) \\ 1(2^3) & 3(2^3.3^1) & 3(2^2.3^2) & 1(3^3) \end{array}$$

Similarly, for $(23)^4$

$$\overset{(2)^4}{16} \xrightarrow{\times\frac{3}{2}} 24 \xrightarrow{\times\frac{3}{2}} 36 \xrightarrow{\times\frac{3}{2}} 54 \xrightarrow{\times\frac{3}{2}} \overset{(3)^4}{81}$$

As Tirthaji shows in his book, addition of 3×24 , 5×36 and 3×54 to the two middle terms gives

$$\mathbf{1}(16) \quad \mathbf{4}(24) \quad \mathbf{6}(36) \quad \mathbf{4}(54) \quad \mathbf{1}(81)$$

which yields the answer 279841 as shown on the previous page.

Thus the usage of Anurupyena (multiplying successive terms by a constant <u>proportion</u> of *b/a*) for a two digit number *ab*, to obtain a power of the number, is seen to be an application of the binomial theorem.

The Yavadunam sutra and the Binomial Theorem

The binomial theorem also underlies the working of the Yavadunam sutra (applied to squaring and higher powers) discussed by Tirthaji in his book:

The Yavadunam ("deficiency") sutra: "Whatever the extent of the deficiency of a number from a base (a power of 10), lessen the number by that very extent; then add the square of the deficiency."

An example:

To obtain 98^2 : Deficiency from 100 is 2.

$98 - 2 = 96$

$2^2 = 04$ Thus $98^2 = 96/04 = 9604$

The method is based on the binomial theorem, because:

$$(100 - x)^2 = \mathbf{1}(100)^2 - \mathbf{2}(100)(x) + \mathbf{1}(x)^2$$

Take out 100 as a common factor from the first two terms on the RHS:

$$\begin{aligned}(100 - x)^2 &= (100)[\mathbf{1}(100) - \mathbf{2}x] + \mathbf{1}(x)^2 \\ &= (100)[(100 - x) - x] + (x)^2\end{aligned}$$

Thus for 98^2 :

$$98^2 = (100-2)^2$$
$$= (100)[(100-2) - 2] + 2^2$$
$$= (100)[98 - 2] + 2^2$$
$$= (100)[96] + 2^2$$
$$= 9600 + 4$$
$$= 9604$$

Use of the deficiency from 98 (in this case 2), is far easier than applying the binomial theorem directly on 98^2 (i.e. $1(90)^2 + 2(90)(8) + 1(8)^2$).

Use of Yavadunam (in conjunction with the binomial theorem), on cubing and higher powers works in the same way. For example:

To obtain 98^3 :

Firstly, use the fact that:

$$(100-x)^3 = \mathbf{1}(100)^3 - \mathbf{3}(100)^2(\boldsymbol{x}) + \mathbf{3}(100)(x)^2 - \mathbf{1}(x)^3$$

Take out 100^2 as a common factor from the first two terms on the RHS:

$$(100-x)^3 = (100)^2[\mathbf{1}(100) - \mathbf{3}x] + (100)(3x^2) - (x)^3$$
$$= (100)^2[(100-x) - \mathbf{2}x] + (100)(3x^2) - (x)^3$$

Thus for 98^3 :

$$98^3 = (100-2)^3$$
$$= (100)^2[(100-2) - \mathbf{2}(2)] + (100)(3)(2)^2 - (2)^3$$
$$= (100)^2[98 - 4] + (100)(12) - 8$$
$$= (10000)(94) + 1200 - 8$$
$$= 940000 + 1200 - 8$$
$$= 941192$$

For cubin, Tirthaji effectively instructs, "Take away *twice* the deficiency, then again add three times the square of the deficiency. Then subtract the cube of the deficiency." This is exactly what was done above.

Thus 98 – $2(2) = 94$ and $3(2^2) = 12$; then add the complement $2^3 = 8$

Thus $94/12/\overline{08} = 94/11/92 = 941192$.

For powers higher than 3, the process becomes more involved, as more and more terms need to be taken into account.

For power 4, *three times* the deficiency is initially subtracted; for power 5, *four times* the deficiency is subtracted, etc. Then the other terms also need to be accounted for.

For example, to obtain 98^4 :

$$(100 - x)^4 \; = \; \mathbf{1}(100)^4 - \mathbf{4}(100)^3(x) + \mathbf{6}(100)^2(x)^{\mathbf{2}} - \mathbf{4}(100)(x)^3 + \; \mathbf{1}(x)^4$$

This can be rewritten as:

$$(100 - x)^4 \; = (100)^2[\mathbf{1}(100) - \mathbf{3}x] + (100)(3x^2) - \; (x)^3$$

$$= (100)^3[(100 - x) - \mathbf{3}x] + \; (100)^2(\mathbf{6}x^2) - (100)(\mathbf{4}x^3) + \; (x)^4$$

Thus for 98^4 :

Thus $98 - 3(2) \; = \; 92$ and $6(2^2) = 24$ and $4(2^3) = 32$ and $2^4 \; = \; 16$

Thus $92/24/\overline{32}/16 \quad = \; 92/23/68/16 \quad = 92236816$

Powering Numbers Above a Base 10

Finding the answer to, say, 106^4 uses a similar algorithm, except for the fact that no subtraction of terms occurs, i.e.

Here we use the fact that $106 = 100 + 6$

Thus $106 + \; 3(6) \; = \; 124$ and $6(6^2) = 216$ and $4(6^3) = 864$ and $6^4 \; = \; 1296$

Thus $124/216/864/1296$

The base here is 100, thus only *two* digits can occur in each term. When digits are correctly carried over to the left we obtain:

$106^4 \quad = \; 124^2/16^8/64^{12}/96 \quad = \; 126/24/76/96 \quad = 126247696$

Using a Working Base

Tirthaji explains the use of working bases on squaring. Working bases can also be applied to finding higher powers.

Example of using a working base of 50 for squaring:

For 48^2 : Using Yavadunam: $48 = 50 - 2$

$48 - 2 \; = \; 46$ and $2^2 \; = \; 4$ This yields $46/04$

Now divide 46 by 2 (as 50 = 100/2). We then obtain $48^2 \; = 2304$

The reason this works is:

$$\begin{aligned} 48^2 &= (50 - 2)^2 \\ &= 50^2 - 2(50)(2) + 2^2 \\ &= 50(50 - 2(2)) + 2^2 \\ &= 50(48 - 2) + 2^2 \\ &= (\tfrac{100}{2})(46) + 4 \end{aligned}$$

$$= (100)\left(\frac{46}{2}\right) + 4$$

$$= 2300 + 4 \qquad = 2304$$

Example of using a working base of 50 for cubing:

For 48^3 : Using Yavadunam: $48 = 50 - 2$

$48 - 2(2) \ = \ 44$ and $(3)(2)^2 \ = \ 12$ while $2^3 = 8$

We have thus far: $44/12/\overline{08}$

The 44 needs to be divided by $2^2 = 4$ (as $(50)^2 = (\frac{100}{2})^2 = \frac{100^2}{4}$) while the 12 needs to be divided by 2 (as $50 = \frac{100}{2}$).

Thus $48^3 = \frac{44}{4}/\frac{12}{2}/\overline{08} \quad = 11/06/\overline{08} \quad = 110592$

Example of using a working base (WB) of 500 for cubing:

For 512^3 : Using Yavadunam: $512 = 500 + 12$

$512 + 2(12) \ = \ 536$ and $(3)(12)^2 \ = \ 432$ while $12^3 = 1728$

We have thus far: $536/432/1728$

The 536 needs to be divided by $2^2 = 4$, while the 432 needs to be divided by 2.

Furthermore, because the WB = 500 = $\frac{1000}{2}$, *three* digits must remain in each term of the calculation.

Thus $512^3 = \frac{536}{4}/\frac{432}{2}/1728 \quad = 134/216^1/728 \quad = 134217728$

Application of Anurupyena and Yavadunam on *Power Triangles*

$$\begin{array}{ccccccccccccc}
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}$$

In the special case of 11 (=10+1), the number formed by the string of digits in the n-th row of the Pascal triangle (with "11" at its pinnacle) is either equal to the *n*-th power of 11, or can be manipulated to equal (by successively carrying a "ten's digit" to the left and adding) the *n*-th power of 11. For instance,

$$(11)^4 = 14641$$

But for $(11)^5$: 1 5 10 10 5 1 becomes $161051 \quad = \quad (11)^5$

And for $(11)^6$: 1 6 15 20 15 6 1 becomes $1771561 \quad = \quad (11)^6$

The question is can the successive powers of *any* two digit number *ab* (other than 11) be obtained by generating a Pascal-type triangle with *ab* at its pinnacle. Such a *Power triangle* for the number 25 is illustrated below:

$$\begin{array}{c}
2 \quad 5 \\
2 \quad 7 \quad 5 \\
2 \quad 9 \quad 12 \quad 5 \\
2 \quad 11 \quad 21 \quad 17 \quad 5 \\
2 \quad 13 \quad 32 \quad 38 \quad 22 \quad 5 \\
2 \quad 15 \quad 45 \quad 70 \quad 60 \quad 27 \quad 5
\end{array}$$

When "tens digits" are carried over to the left and added, the following numbers are obtained in the successive rows:

Power Triangle for 25

$n = 1$	2 5	$= P_1$	$P_1 = 25 \times (11)^0$
$n = 2$	2 7 5	$= P_2$	$P_2 = 25 \times (11)^1$
$n = 3$	3 0 2 5	$= P_3$	$P_3 = 25 \times (11)^2$
$n = 4$	3 3 2 7 5	$= P_4$	$P_4 = 25 \times (11)^3$
$n = 5$	3 6 6 0 2 5	$= P_5$	$P_5 = 25 \times (11)^4$
$n = 6$	4 0 2 6 2 7 5	$= P_6$	$P_6 = 25 \times (11)^5$

We now show how the number in the *n*-th row of this triangle i.e. P_n is given by,

$$P_n = 25 \times (11)^{n-1}$$

Some successive powers of 25 are:

$n = 1$	2 5	$= 25^1$
$n = 2$	6 2 5	$= 25^2$
$n = 3$	1 5 6 2 5	$= 25^3$
$n = 4$	3 9 0 6 2 5	$= 25^4$
$n = 5$	9 7 6 5 6 2 5	$= 25^5$
$n = 6$	2 4 4 1 4 0 6 2 5	$= 25^6$

Thus the numbers formed in each row of the "Pascal power triangle" for 25 do not directly yield powers of 25.

The numbers in the power triangle can, however, be manipulated (by use of Anurupyena) to, indeed, yield the powers of 25. The process is shown below:

For example, to calculate the value of $(25)^4$ (thus $n = 4$)

Let $x = 2$ and $y = (5 - 2) = 3$. The constant multiplier is $\frac{y}{x} = \frac{3}{2}$.

Use Anurupyena to generate the coefficients for $\underline{n = 3}$ ($\underline{\text{not}}$ $n = 4$)

$$\overset{(2)^3}{8} \xrightarrow{\times\frac{3}{2}} 12 \xrightarrow{\times\frac{3}{2}} 18 \xrightarrow{\times\frac{3}{2}} \overset{(3)^3}{27}$$

$$\begin{array}{cccc} 8 & 12 & 18 & 27 \\ & 24 & 36 & \\ \hline 8 & 36 & 54 & 27 \end{array}$$

These coefficients (8, 36, 54 and 27) are then respectively multiplied with the numbers in rows $n = 4$ to $n = 1$ (i.e. P_4, P_3, P_2 and P_1, see below) in the power triangle for 25. When these terms are added together, the 4th power of 25 is obtained:

$$\begin{array}{lcccccccl} n = 1 & & & & & 2 & 5 & & = P_1 \\ n = 2 & & & & 2 & 7 & 5 & & = P_2 \\ n = 3 & & & 3 & 0 & 2 & 5 & & = P_3 \\ n = 4 & & 3 & 3 & 2 & 7 & 5 & & = P_4 \\ n = 5 & 3 & 6 & 6 & 0 & 2 & 5 & & = P_5 \\ n = 6 & 4\;\,0 & 2 & 6 & 2 & 7 & 5 & & = P_6 \end{array}$$

Thus $(25)^4 = 8(33275) + 36(3025) + 54(275) + 27(25) = \underline{390625}$

Or, written in terms of the binomial formulation:

$$\begin{aligned} (25)^4 &= \mathbf{1}(2)^3(33275) + \mathbf{3}(2^2)(5-2)^1(3025) + \mathbf{3}(2^1)(5-2)^2(275) + \mathbf{1}(5-2)^3(25) \\ &= \mathbf{1}(2)^3(33275) + \mathbf{3}(2^2)(3)^1(3025) + \mathbf{3}(2^1)(3)^2(275) + \mathbf{1}(3)^3(25) \\ &= \mathbf{1}(2)^3(33275) + \mathbf{3}(12)(3025) + \mathbf{3}(18)(275) + \mathbf{1}(3)^3(25) \end{aligned}$$

In general, to obtain the n-th power of any two digit number ab:

Generate successive coefficients, using the multiplier $\frac{(b-a)}{a}$:

$$a^{n-1} \xrightarrow{\times\frac{(b-a)}{a}} a^{n-2}(b-a) \xrightarrow{\times\frac{(b-a)}{a}} a^{n-3}(b-a)^2 \xrightarrow{\times\frac{(b-a)}{a}} a^{n-4}(b-a)^3 \ldots \xrightarrow{\times\frac{(b-a)}{a}} \ldots (b-a)^{n-1}$$

These coefficients are then multiplied with their respective binomial coefficients, as well as, respectively, with the numbers in rows $n,\ n-1,\ n-2, \ldots 3,\ 2,\ 1$ in the power triangle for ab. When these terms are added together, the nth power of ab is obtained:

Thus $(ab)^n$ can be found using

$$\binom{n-1}{0} a^{n-1}(P_n) + \binom{n-1}{1} a^{n-2}(b-a)^1(P_{n-1}) + \binom{n-1}{2} a^{n-3}(b-a)^2(P_{n-2}) + \ldots\ldots + \binom{n-1}{n-2} a^1(b-a)^{n-2}(P_2) + \binom{n-1}{n-1}(b-a)^{n-1}(P_1)$$

(where P_n is the number in the n-th row)

Why does the Method Work?

For the general case, the power triangle for ab is:

$$a \quad b$$

$$a \quad (a+b) \quad b$$

$$a \quad (2a+b) \quad (a+2b) \quad b$$

$$a \quad (3a+b) \quad (3a+3b) \quad (a+3b) \quad b$$

$$a \quad (4a+b) \quad (6a+4b) \quad (4a+6b) \quad (a+4b) \quad b$$

$$a \quad (5a+b) \quad (10a+5b) \quad (10a+10b) \quad (5a+10b) \quad (a+5b) \quad b$$

The actual two digit number ab can be represented by $10a+b$. Taking into account the carrying to the left of tens, hundreds, etc. the power triangle for $10a+b$ is thus:

$$10a+b$$

$$100a+10(a+b)+\ b$$

$$1000a+100(2a+b)+10(a+2b)+\ b$$

$$10000a+1000(3a+b)+100(3a+3b)+10(a+3b)+\ b$$

$$100000a+10000(4a+b)+1000(6a+4b)+100(4a+6b)+\ 10(a+4b)+b$$

$$1000000a+100000(5a+b)+10000(10a+5b)+1000(10a+10b)+\ 100(5a+10b) + \ 10(a+5b)+b$$

This reduces to:

$10a+b \quad = \quad 1(10a+b) \quad = \quad 11^0(10a+b) \quad = P_1$

$110a+11b \quad = \quad 11(10a+b) \quad = \quad 11^1(10a+b) \quad = P_2$

$1210a+121b \quad = \quad 121(10a+b) \quad = \quad 11^2(10a+b) \quad = P_3$

$13310a+1331b \quad = \quad 1331(10a+b) \quad = \quad 11^3(10a+b) \quad = P_4$

$146410a+14641b \quad = \quad 14641(10a+b) \quad = \quad 11^4(10a+b) \quad = P_5$

$1610510a+161051b \quad = \quad 161051(10a+b) \quad = \quad 11^5(10a+b) \quad = P_6$

We thus see that the number P_n in the n-th row of the reduced power triangle for ab always equals:

$$P_n = (10a+b)(11)^{n-1}$$

The numbers in each row thus form a geometric sequence with first term $(10a+b)$ (at the pinnacle of the power triangle) and constant ratio 11.

For the case of $(ab)^4$, application of the procedure:

$$\binom{n-1}{0}a^{n-1}(P_n)+\binom{n-1}{1}a^{n-2}(b-a)^1(P_{n-1})+\binom{n-1}{2}a^{n-3}(b-a)^2(P_{n-2})+\dots\dots$$
$$+\binom{n-1}{n-2}a^1(b-a)^{n-2}(P_2)+\binom{n-1}{n-1}(b-a)^{n-1}(P_1)$$

$$=\mathbf{1}(a)^3(P_4)+\mathbf{3}(a^2)(b-a)^1(P_3)+\mathbf{3}(a^1)(b-a)^2(P_2)+\mathbf{1}(b-a)^3(P_1)$$

$$=\mathbf{1}(a)^3(11)^3(10a+b)+\mathbf{3}(a^2)(b-a)^1(11)^2(10a+b)+\mathbf{3}(a^1)(b-a)^2(11)^1(10a+b)+\mathbf{1}(b-a)^3(11)^0(10a+b)$$

$$=(10a+b).[(a)^3(11)^3+\mathbf{3}(a^2)(b-a)^1(11)^2+\mathbf{3}(a^1)(b-a)^2(11)^1+\mathbf{1}(b-a)^3(11)^0]$$

$$=(10a+b).[(11^3-3.11^2+3.11-1)a^3+(3.11^2-6.11+3)a^2b+(3.11-3)ab^2+b^3]$$

$$=(10a+b).[\mathbf{1}(11-1)^3a^3+\mathbf{3}(11-1)^2a^2b+\mathbf{3}(11-1)^1ab^2+\mathbf{1}b^3]$$

$$=(10a+b).[\mathbf{1}(10)^3a^3+\mathbf{3}(10)^2a^2b+\mathbf{3}(10)^1ab^2+\mathbf{1}b^3]$$

$$=(10a+b).[10a+b]^3$$

$$=(10a+b)^4$$

Thus for any power triangle with pinnacle ab (i.e. the number $10a+b$) and with P_n the number in the n-th row:

$$(10a+b)^n=\sum_{k=0}^{n-1}\binom{n-1}{k}a^{(n-1)-k}(b-a)^kP_{n-k}$$

where $P_n=(10a+b)(11)^{n-1}$

An example:

Use a power triangle to calculate 74^3.

7 4

8 1 4

8 9 5 4

Apply Anurupyena using the ratio $\frac{4-7}{7}=\frac{-3}{7}$:

7^2 $(4-7)^2$

$$7^2 \quad \underset{\times\frac{-3}{7}}{\curvearrowright} \quad -21 \quad \underset{\times\frac{-3}{7}}{\curvearrowright} \quad 3^2$$

Now multiply 1(49), 2(-21) and 1(9) respectively with 8954, 814 and 74, and add:

$$74^3=49(8954)-2(21)(814)+9(74)=405224$$

Although this process appears to involve more work than, say, the use of Yavadunam (with base 100), there are certain cases where a power triangle might be useful to help find the answer to the power of a number quickly and efficiently:

Power Triangle for the Case where *a* = *b*

When $a = b$ it is possible to find any power by doing only a single multiplication.

Finding 44^3

$$\begin{array}{c} 4 \quad 4 \\ 4 \quad 8 \quad 4 \\ 5 \quad 3 \quad 2 \quad 4 \\ 5 \quad 8 \quad 5 \quad 6 \quad 4 \end{array} \qquad = 44 \times 11^3$$

In this case, the ratio is $\frac{4-4}{4} = 0$. Thus all the terms, except the first term, become zero.

$$4^3 \xrightarrow{\times 0} 0 \xrightarrow{\times 0} 0 \xrightarrow{\times 0} (4-4)^3$$

Thus $44^3 = 4^2(5324) + 0(484) + 0(44)$

$$= 16 \times 5324 = 85184$$

Similarly, $44^4 = 4^3(58564) = 64 \times 58564 = 3748096$

Note: Another explanation for this phenomenon is the fact that e.g.

$$\begin{aligned} 44^4 &= (4 \times 11)^4 \\ &= (4 \times 11) \times (11)^3 \times (4)^3 \\ &= (44 \times 11^3) \times (4)^3 \\ &= (58564) \times (64) \end{aligned}$$

Another example:

Finding 33^6

$$\begin{array}{c} 3 \quad 3 \\ 3 \quad 6 \quad 3 \\ 3 \quad 9 \quad 9 \quad 3 \\ 4 \quad 3 \quad 9 \quad 2 \quad 3 \\ 4 \quad 8 \quad 3 \quad 1 \quad 5 \quad 3 \\ 5 \quad 3 \quad 1 \quad 4 \quad 6 \quad 8 \quad 3 \end{array}$$

$33^6 = 3^5 \times 5314683 = 243 \times 5314683 = 1291467969$

The Complement of a Power Triangle

When the sutra Nikhilam Navatascaraman Dasatah (All from 9 and the last from 10) is applied to the number at the pinnacle of any power triangle, the *complement* of that number is found. The power triangle related to this complement has the following relationship to the original power triangle:

$$P_n + P_n' = 100(11^{n-1})$$

where P_n and P_n' are, respectively, the numbers in the n-th rows of the original power triangle and the power triangle of its complement.

E.g. for “11” at the pinnacle, with “89” its complement:

$$P_4 + P_4' = 14641 + 118459 = 133100 = 100(11^3)$$

1 1	8 9
1 2 1	9 7 9
1 3 3 1	1 0 7 6 9
1 4 6 4 1	1 1 8 4 5 9
1 6 1 0 5 1	1 3 0 3 0 4 9
1 7 7 1 5 6 1	1 4 3 3 3 5 3 9

Since $P_4 = 14641 = 11^4$ it follows that

$$P_4' = 100(11^3) - 11^4$$

$$= 11^3(100 - 11)$$

$$= 11^3(89) = 118459$$

The addition of each respective row in the above two power triangles thus leads to:

1 0 0

1 1 0 0

1 2 1 0 0

1 3 3 1 0 0

1 4 6 4 1 0 0

1 6 1 0 5 1 0 0

This property of summation of rows is reminiscent of the multiplication of a matrix by its inverse matrix (to obtain the identity matrix), and might have some uses in helping to find powers of numbers in a quicker way.

This property applies to all power triangles with pinnacle “ab”, e.g.

2 3		7 7		1 0 0
2 5 3		8 4 7		1 1 0 0
2 7 8 3	+	9 3 1 7	=	1 2 1 0 0
3 0 6 1 3		1 0 2 4 8 7		1 3 3 1 0 0
3 3 6 7 4 3		1 1 2 7 3 5 7		1 4 6 4 1 0 0

Using Derivatives and Integrands to Find Binomial Terms

For a binomial of the form $(x+y)^n$, if D_n is the n-th derivative of x^n with respect to x and I_n is the n-th integrand of y^0 with respect to y, then the expansion is:

$$(x+y)^n \quad = x^n y^0 + D_1(x^n)I_1(y^0) + D_2(x^n)I_2(y^0) + \cdots + D_n(x^n)I_n(y^0)$$

which can be abbreviated as:

$$(a+b)^n = \sum_{r=0}^{n} D_r(a^n)I_r(b^0)$$

Suppose you wish to expand $(a+b)^4$:

The first step is to set down the first term, a^4b^0.

The second term is derived by differentiating a^4 with respect to a, and integrating b^0 with respect to b. This gives $4a^3 \times \frac{b^1}{1}$.

The third term is found by differentiating $4a^3$ with respect to a, and integrating $\frac{b^1}{1}$ with respect to b. This gives $3\times 4a^2 \times \frac{b^2}{1\times 2}$.

Similarly, successive differentiation and integration gives the whole sequence as:

$$(a+b)^4$$

$$= (a^4b^0) + \left(4a^3 \times \frac{b^1}{1}\right) + \left(3\times 4a^2 \times \frac{b^2}{1\times 2}\right) + \left(2\times 3\times 4a^1 \times \frac{b^3}{1\times 2\times 3}\right)$$
$$+ (1\times 2\times 3\times 4a^0 \times \frac{b^4}{1\times 2\times 3\times 4})$$

which simplifies to:

$$(a+b)^4 = a^4 + 4a^3b^1 + 6a^2b^2 + 4a^1b^3 + 1b^4$$

This may seem a cumbersome method for obtaining a simple expansion, but its advantages become obvious when dealing with expansions involving fractional and negative powers. Many students find that, when learning the binomial formula for the first time, they experience difficulty in remembering it and how it works. This calculus method is very easy to remember and easy to apply.

Example 1: Expand $(2+x)^{-3}$, first four terms.

$$(2+x)^{-3} \ = \ (2^{-3}x^0) + (-3.2^{-4}\times x^1) + \left(-4.-3.2^{-5}\times\frac{x^2}{2}\right) + \left(-5.-4.-3.2^{-6}\times\frac{x^3}{3.2}\right) + \cdots$$

$$= \tfrac{1}{8} - \tfrac{3}{16}x + \tfrac{3}{16}x^2 - \tfrac{5}{32}x^3 + \ \ldots$$

Example 2: Expand $(1+2x)^{1/2}$, up to and including the term in x^3.

$$(1+2x)^{1/2} \ = \ 1^{1/2}(2x)^0 + \tfrac{1}{2}.1^{-1/2}(2x)^1 + -\tfrac{1}{2}.\tfrac{1}{2}.1^{-3/2}\tfrac{(2x)^2}{2} + -\tfrac{3}{2}.-\tfrac{1}{2}.\tfrac{1}{2}.1^{-5/2}\tfrac{(2x)^3}{2\times 3} + \ \ldots$$

$= 1 + x - \frac{1}{2}x^2 + \frac{1}{2}x^3 + \ldots$

Example 3: Find the first four terms, in ascending powers of x, of $(9-x)^{1/2}$

$$(9-x)^{1/2} = 9^{1/2}(-x)^0 + \tfrac{1}{2}.9^{-1/2}(-x)^1 + -\tfrac{1}{2}.\tfrac{1}{2}.9^{-3/2}\frac{(-x)^2}{2} + -\tfrac{3}{2}.-\tfrac{1}{2}.\tfrac{1}{2}.9^{-5/2}\frac{(-x)^3}{2\times 3} + \ldots$$

$$= 3 - \frac{1}{6}x - \frac{1}{216}x^2 - \frac{1}{3888}x^3 + \ldots$$

(Note that there is no need to make the first term of the binomial equal to 1.)

Proof that Successive Terms of the Binomial Expansion are Obtained by Differentiation and Integration

The $(r+1)th$ term in the binomial expansion of $\left(a+b\right)^n$ is $\dfrac{n!}{r!(n-r)!}a^{n-r}b^r$.

Differentiating a^{n-r} and integrating b^r result in

$$\frac{n!(n-r)a^{n-r-1}b^{r+1}}{r!(n-r)!(r+1)} = \frac{n!a^{n-r-1}b^{r+1}}{(r+1)!(n-r-1)!} = \frac{n!a^{n-(r+1)}b^{r+1}}{(r+1)!(n-(r+1))!}$$

This is the $(r+2)th$ term.

In general, let $D_r(a^n)$ be the *r*-th derivative of a^n with respect to *a*. Let $I_r(b^0)$ be the *r*-th integral of b^0 with respect to *b*.

Then the binomial expansion can be summarised as follows:

$$\left(a+b\right)^n = \sum_{r=0}^{r=n} D_r(a^n)I_r(b^0)$$

Alternative Approach: Just using Differentiation

It is possible to achieve the same result as discussed above by successively differentiating the first term (a) in the binomial from *left to right*, whilst at the same time successively differentiating the second term (b) in the binomial from *right to left*. The sum of the products of the successive terms (divided by $n!$) also yield the binomial expansion, i.e.

$$(a+b)^n = \frac{1}{n!}\sum_{r=0}^{n} D_r(a^n)D_{n-r}(b^n)$$

(This formula can also be proved from a Maclauren series. See last page.)

E.g. $(a+b)^4 = \frac{1}{4!}\sum_{r=0}^{4} D_r(a^4)D_{4-r}(b^4)$

$= \frac{1}{4!}[D_0(a^4)D_4(b^4) + D_1(a^4)D_3(b^4) + D_2(a^4)D_2(b^4) + D_3(a^4)D_1(b^4) + D_4(a^4)D_0(b^4)]$

$$= \frac{1}{4!}[a^4.1.2.3.4b^0 + 4a^3.2.3.4b^1 + 3.4a^2.3.4b^2 + \ 2.3.4a^1.4b^3 + 1.2.3.4a^0.b^4\ ]$$

$$= \frac{1}{4!}\left[4!\,a^4 + 4!\,4a^3b^1 + 4!\frac{3.4}{2}a^2b^2 + \ 4!\,4a^1b^3 + 4!\,a^0.b^4\ \right]$$

$$= a^4 + 4a^3b^1 + 6a^2b^2 + 4a^1b^3 + 1b^4$$

Example 4:

Find $(23)^4$ using the $D_r(a^n)I_r(b^0)$ method.

"Differentiate" 2^4 four times:	2^4	4.2^3	12.2^2	24.2^1	24.2^0
"Integrate" 3^0 four times:	3^0	$\frac{3^1}{1}$	$\frac{3^2}{2}$	$\frac{3^3}{2.3}$	$\frac{3^4}{2.3.4}$
Multiply successive terms	2^4	$4.8.3$	$12.4.\frac{9}{2}$	$24.2.\frac{27}{6}$	$24.\frac{81}{24}$
Simplify	16	96	216	216	81

Carry "hundreds" and "tens and the answer is 279841.

Example 5:

Find $(23)^4$ using the $D_r(a^n)D_{n-r}(b^n)$ method.

"Differentiate" 2^4 four times, left to right:	2^4	4.2^3	12.2^2	24.2^1	24.2^0
"Differentiate" 3^4 four times, right to left:	24.3^0	24.3^1	12.3^2	4.3^3	3^4
Multiply successive terms	24.2^4	$24.3.4.2^3$	$12.3^2.12.2^2$	$24.2.4.3^3$	24.3^4
Divide each term by $4! = 24$	2^4	$3.4.2^3$	$3^2.6.2^2$	$2.4.3^3$	3^4
Simplify	16	96	216	216	81

Carry "hundreds" and "tens" 279841

Derivation from Maclaurin Series

For $f(a) = (a+b)^n$ let a be a variable, and b a constant.

$$f(a) = (a+b)^n \quad \rightarrow \quad f(0) = b^n$$

$$f'(a) = n(a+b)^{n-1} \quad \rightarrow \quad f'(0) = nb^{n-1}$$

$$f''(a) = n(n-1)(a+b)^{n-2} \quad \rightarrow \quad f''(0) = n(n-1)b^{n-2}$$

$$f'''(a) = n(n-1)(n-2)(a+b)^{n-3} \quad \rightarrow \quad f'''(0) = n(n-1)(n-2)b^{n-3}$$

$$f^{n-3}(a) = n(n-1)(n-2)\dots(4)(a+b)^3 \quad \rightarrow \quad f^{n-3}(0) = n(n-1)(n-2)\dots(4)b^3$$

$$f^{n-2}(a) = n(n-1)(n-2)\dots(3)(a+b)^2 \quad \rightarrow \quad f^{n-2}(0) = n(n-1)(n-2)\dots(3)b^2$$

$$f^{n-1}(a) = n(n-1)(n-2)\dots(2)(a+b)^1 \quad \rightarrow \quad f^{n-1}(0) = n(n-1)(n-2)\dots(2)b^1$$

$$f^{n}(a) = n(n-1)(n-2)\dots(1)(a+b)^0 \quad \rightarrow \quad f^{n}(0) = n(n-1)(n-2)\dots(1)b^0$$

Maclauren Series:

$$f(a) = f(0)a^0 + \frac{f'(0)}{1!}a^1 + \frac{f''(0)}{2!}a^2 + \frac{f'''(0)}{3!}a^3 + \cdots + \frac{f^{n-3}(0)}{(n-3)!}a^{n-3} + \frac{f^{n-2}(0)}{(n-2)!}a^{n-2} + \frac{f^{n-1}(0)}{(n-1)!}a^{n-1} + \frac{f^{n}(0)}{n!}a^{n}$$

$$f(a) = b^n a^0 + \frac{nb^{n-1}}{1!}a^1 + \frac{n(n-1)b^{n-2}}{2!}a^2 + \frac{n(n-1)(n-2)b^{n-3}}{3!}a^3 + \cdots + \frac{n(n-1)(n-2)\dots(4)b^3}{(n-3)!}a^{n-3} + \frac{n(n-1)(n-2)\dots(3)b^2}{(n-2)!}a^{n-2} + \frac{n(n-1)(n-2)\dots(2)b^1}{(n-1)!}a^{n-1} + \frac{n!\,b^0}{n!}a^n$$

$$f(a) = \frac{1}{n!}\left[\ (n!\,a^0)(b^n) + \left(\frac{n!a^1}{1!}\right)(nb^{n-1}) + \left(\frac{n!a^2}{2!}\right)(n(n-1)b^{n-2}) + \left(\frac{n!a^3}{3!}\right)(n(n-1)(n-2)b^{n-3}) + \cdots + \left(\frac{n!a^{n-3}}{(n-3)!}\right)(n(n-1)(n-2)\dots(4)b^3) + \left(\frac{n!a^{n-2}}{(n-2)!}\right)(n(n-1)(n-2)\dots(3)b^2) + \left(\frac{n!a^{n-1}}{(n-1)!}\right)(n(n-1)(n-2)\dots(2)b^3) + (a^n)(n!\,b^0)\ \right]$$

$$f(a) = \frac{1}{n!}\ [D_n(a^n)D_0(b^n) + D_{n-1}(a^n)D_1(b^n) + D_{n-2}(a^n)D_2(b^n) + D_{n-3}(a^n)D_3(b^n) + \cdots + D_3(a^n)D_{n-3}(b^n) + D_2(a^n)D_{n-2}(b^n) + D_1(a^n)D_{n-1}(b^n) + D_0(a^n)D_n(b^n)\ ]$$

Conclusion and Comments

With regards to the application of the binomial theorem to cubing and higher powers, Tirthaji comments: "Almost every mathematical worker *knows* this (the binomial theorem); but very few people *apply* it! (e.g. when finding the power of an actual number.) This is the whole tragedy and pathos of the situation."

Many students today learn how to use the binomial theorem in algebra and statistics courses, but never actually think to apply it also to basic arithmetic. This is exactly what the Anurupyena and Yavadunam sutras achieve when they are used to calculate the squares, cubes and higher powers of numbers. The answer is obtained with ease and swiftness, a fundamental feature of the Vedic mathematical approach.

The same can be said for the differentiation/integration approach to obtaining a binomial expansion: Maclauren's (and Taylor's) theorems are conventionally employed as methods to approximate a function as a polynomial with a finite or infinite number of terms. However, as has been illustrated, it is possible to obtain the *n*-th power of an actual *number* by successive “differentiations” and/or” integrations” of its digits. This is, again, in line with the Vedic mathematical approach of obtaining an answer in “one line” (left to right or right to left).

The use of Pascal’s arithmetic triangle, as well as the related power triangle, is also a useful tool to help simplify calculations. There are possibly more properties of power triangles which are worth investigating.

There is evidence that the arithmetic triangle has been known and used for hundreds, if not thousands of years, before Pascal wrote about it in the 1600’s. Pingala, an Indian mathematician and Sanskrit grammarian, wrote about the arithmetic triangle and the binomial theorem in approximately 200BC. (Pingala’s *Chanda-shastrahas* been recorded as the most ancient Sanskrit treatise on prosody.) In the Chinese book *Xiangjie Jiuzhang Suanfa* (1261 AD), Yang Hui records the work of mathematician Jia Xian who also illustrated such a triangle about 500 years before Pascal.

Refernce

Vedic Mathematics, B.K. Tirthaji, Motilal Banarsidass, 1965