SOLUTION OF RIGHT-ANGLED TRIANGLES USING *VERTICALLY AND CROSSWISE*

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Abstract

This paper shows how the *Vertically and Crosswise* Sutra of Vedic Mathematics[1] can be used to approximate sides and angles (in degree measure) of right-angled triangles without resorting to a calculator, tables etc.

Knowing just one triangle, with sides 3, 4, and 5 units (and angles 37°, 53° and 90°, to the nearest degree), and knowing how to handle triangles with a small angle, we can estimate sides and angles of any other triangle. And knowing more such 'Pythagorean Triples' like 3,4,5 we can improve accuracy as far as required.

1. Introductory

The calculation of sides and angles of triangles is the first instance where schoolchildren are expected to take values for granted. The teacher, and therefore the pupils, accept that a calculator, tables or other device is essential to find of sines, cosines etc. and no indication is usually given as to how they may be derived.

For accurate and fast results the calculator is a marvellous and essential device, but it would be more educationally sound for pupils to have some easy way to find these values without a calculator, even if only with limited accuracy.

1.1 Why Teach Solution of Triangles without a Calculator?

Trigonometric functions are particularly difficult to calculate, and even many teachers might be hard pressed to say how they may be evaluated. Consequently no one ever calculates trig functions and their inverses as it is considered too hard − we resort to a calculator, tables or other method. Why should we bother to calculate trig functions when they are so easily available from a calculator?

The same question can be asked about square roots, division, multiplication . . . So where would we stop and say these should be taught with pencil and paper and these not?

The reason for teaching the calculation of products and quotients is so that pupils get to understand better what they mean, and so that in the absence of a calculator, or for quick on-the-spot calculations or estimates, we can get results.

But these same reasons apply to the solution of right-angled triangles, which are also of great practical use.

We may also note that in many countries, calculators are not available to children.

Before showing an example of this method (in Section 2) some things can usefully be explained: on handling proportion and on triples, triple addition and small angles.

1.2. Proportion

We frequently need to handle proportional questions and these can simplified with the following arrangement.

Given two similar triangles for example with sides *a*, *b*, *c* and *A*, *B*, *C* we arrange them with their corresponding sides aligned:

a	b	c
A	B	C

Then if we have a triangle with sides 2, 3, 4, say, and require the length of the 3rd element in a similar triangle where the 2nd element is 19:

2	3	4
–	19	x

We find x by cross-multiplication: $x = \frac{4\times 19}{3}$.

Such an arrangement can be used for all kinds of proportion (direct, inverse, inverse square etc.), with 2, 3 or more elements and will be indicated by a bordered box in this paper.

1.3. Triples

We call a set of three real numbers a, b, c a triple if $a^2 + b^2 = c^2$.

So a, b, c can represent the sides of a right-angled triangle.

Further, we use the notation A) a, b, c when we wish to include the angle. And the angle referred to is between the side of length a (the base) and the side of length c (the hypotenuse).

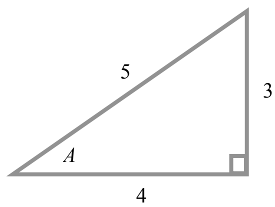


Figure 1

Thus A) 4, 3, 5 would represent the triangle shown in Figure 1.

Of course the base may not be horizontal: it could be vertical or at an oblique angle.

Triple addition and subtraction are defined by:

A	a	b	c	
B	d	e	f	$\pm$
$A+B$	$ad-be$	$bd+ae$	cf	
$A-B$	$ad+be$	$bd-ae$	cf	

The bottom two lines in the above chart follow immediately from the standard expansions for $\cos(A \pm B)$ and $\sin(A \pm B)$ and may also be proved by similar triangles (i.e. by simple proportion).

1.4. Small Triples

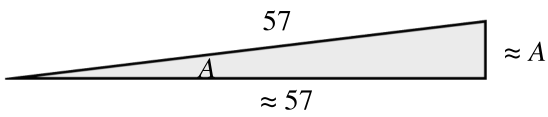
A *small triple* is defined as a triple in which the angle is small, say 0° to 10°.

A small triple can be described approximately by A) 57, A, 57 where A is a small angle in degrees and 57 is an approximation to 180/π.

This is explained by the fact that if we enquire what radius a circle should have such that any arc length is equal to the angle, in degrees, which it subtends at the centre the answer is 180/π (i.e. approximately 57).

So if the angle in a triangle is small then the base and hypotenuse may be considered close and the height approximately equal to the angle.

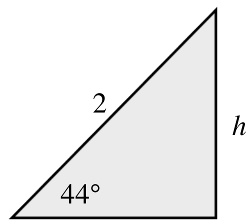
For example 7°)57, 7, 57 is a small triple for an angle of 7°.



2. Finding a Side

We will find that lengths are generally accurate to 2 s.f. but will be given here to 3 s.f., together with the percentage error, e, which will compare the value obtained with the true value.

2.1 'Sine' Type



Example 1a: Solve 44°) –, h, 2

We need to estimate the height h of the triangle shown.

Supposing 44° to be close to the angle in the 37°) 4,3,5 triple we could find an answer using simple proportion:

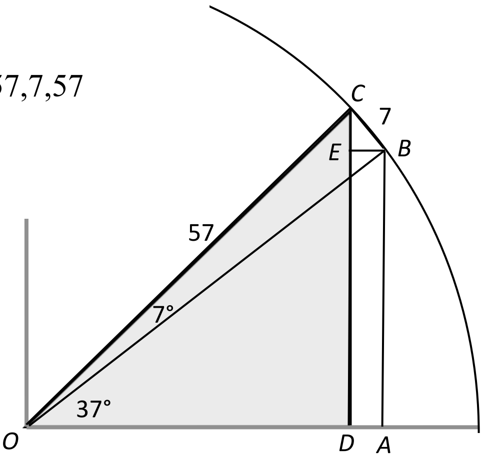
4	3	5
–	h	2

$h = 2 \times \frac{3}{5} = 1.2$.

We may call this a first approximation to h.

For better accuracy we need to take the difference between the angles 37°, 44° into account.

That difference can be expressed by the small triple 7)57,7,57 and triple addition then gives the ratio of height to hypotenuse for a triple with angle 44°:



37°	4	3	5	
7°	57	7	57	+
44°	–	$3\times57+4\times7$	5×57	

And this is similar to the triangle we want, so we use proportion:

37°	4	3	5	
7°	57	7	57	+
44°	–	$3\times57+4\times7$	5×57	
44°	–	h	2	

Thus we find that

$h = 2 \times \frac{3\times57+4\times7}{5\times57}$ = 1.40 to 3 s.f. [e = 0.5%]

And we see from this that we can find h using a simple pattern (see Figure 2):

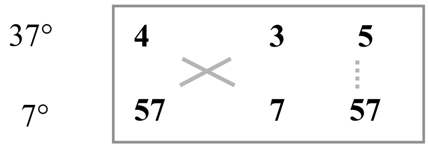


Figure 2

That is:

1) We set up our nearby triple, 37°) 4,3,5.

2) We set up a 'small triple' for the difference between the known angle (37°) and the given angle (44°): 7°)57,7,57.

3) We multiply the given hypotenuse, 2, by a fraction whose numerator is given by the crosswise operation indicated above, and whose denominator is given by the vertical operation indicated.

Note 1: The above derivation of the pattern constitutes a proof as the numbers are distinct and no evaluations have been carried out.

Note 2: And having established the pattern we can use it on all triangles of this type.

Note 3: If we write $h = 2 \times \frac{3\times57+4\times7}{5\times57}$ as $h = 2\times\frac{3}{5} + 2\times\frac{4\times7}{5\times57}$,

then $2\times\frac{3}{5}$ is the first approximation we obtained above, and $2\times\frac{4\times7}{5\times57}$ (equal to CE if the radius in

the diagram above was 2 instead of 57) is the adjustment required to make the original estimate (equivalent to BA) more accurate.

Note 4: The above result can also be arrived at using only *Proportion* (i.e. without invoking the algorithm for triple addition) by consideration of the 'similar' triangles *CEB* and *OAB* (these triangles are similar 'in the limit' i.e. as the small angle tends to zero).

Note 5: This method is reversible so that given a height and hypotenuse we can find the angle (see Section 3).

2.1.1 Simplifying the Calculation

Example 1b: Solve 44°) —, *h*, 2

The use of the number 57 generally makes the calculation awkward, but since the adjustment referred to in Note 3 above is comparatively small we may alter the 57,7,57 to 8,1,8 by changing our 57 to 56 and cancelling by 7. This gives:

37°	**4**	**3**	**5**
7°	**8**	**1**	**8**

and we get $h = 2 \times \frac{3\times8+4\times1}{5\times8} = 1.40$, the same value to 3 s.f. that we had in Example 1a above.

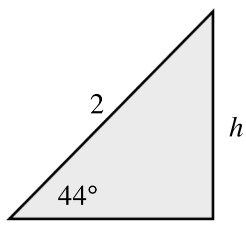
[e = 0.8%]

Note: This simplification considerably reduces the work involved and makes mental solution of this type of problem entirely possible. Other ratios can be used too. For example for 6° we note that 57÷6 = 19/2, so we can use 9,1,9 or 10,1,10, or for greater accuracy 19,2,19.

2.1.2 Negative Angle

Example 1c: Solve 44°) —, *h*, 2

When the given angle is below the known angle the method is just the same, but we use a negative angle.
For example if we were to use 53°) 3,4,5 as the known triple we need to add -9° to get 44°:



53°	**3**	**4**
		5
-9°		

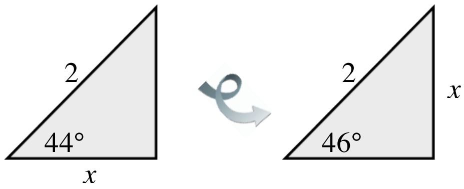
Then the same pattern as before gives the answer: $h = 2 \times \frac{4\times57+3\times(-9)}{5\times57} = 1.41$ to 3 s.f.

[e = 1.5%]

Two further cases are easily solved by the method shown above as Examples 2 and 3 show.

2.2 'Cosine' Type

Example 2: Solve 44°) *x*, —, 2



We can simply switch to the complementary triple (*Transpose and Apply* Sutra):

44°) *x*, —, 2 becomes 46°) —, *x*, 2

Then:

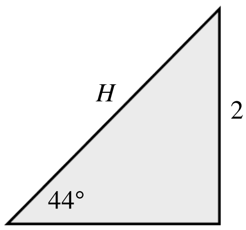
53°	**3**	**4**
	5	
-7°		

And so $x = 2 \times \frac{4\times8+3\times(-1)}{5\times8} = 1.45$ to 3 s.f. [$e = 0.8\%$]

Note: An alternative variation of the given pattern can be used instead of switching to the complementary triple.

2.3 Finding the Hypotenuse

Example 3: Solve 44°) —, 2, *H*



A typical problem is shown opposite.
Below is the addition we had earlier for 44°.

37°	4	3	5	
7°	57	7	57	+
44°	—	3×57+4×7	5×57	

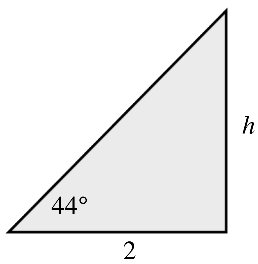
—	3×57 + 4×7	5×57
—	2	*H*

and so $H = 2 \times \frac{5\times57}{3\times57+4\times7} = 2.86$ to 3 s.f. [$e = 0.5\%$]

We note that this calculation is identical to that in Example 1a except that the fraction is now inverted.

2.4 'Tangent' Type

Example 4: Solve 44°) 2, *h*, —



This is illustrated by the diagram opposite.

Once again we use the triple for 44° and *Proportion*:

4×57 – 3×7	3×57 + 4×7	5×57
2	*h*	—

We find $h = 2 \times \frac{3\times57+4\times7}{4\times57-3\times7} = 1.92$ to 3 s.f. [$e = 0.5\%$]

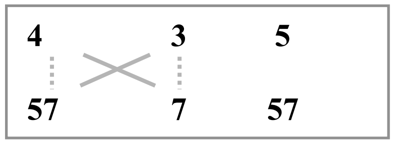
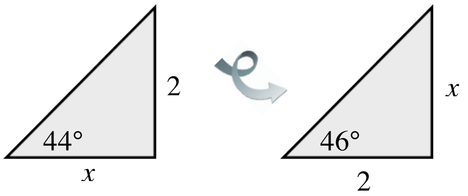


Figure 3

So the pattern here is a bit different (see Figure 3).

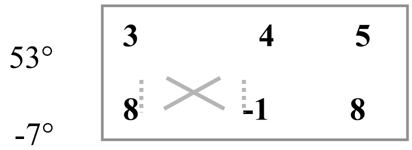
We multiply crosswise and add for the numerator as before,
but the denominator is found by multiplying vertically and subtracting.

Example 5: Solve 44°) x, 2, —



In this case we would use the complementary triple so that, as in Example 4, we have the unknown as a height.

Then:



And so $x = 2 \times \frac{4\times8+3\times(-1)}{3\times8-4\times(-1)} = 2.07$ to 3 s.f. [$e = 0.02\%$]

3. Finding an Angle

3.1 'First Approximation'

As in the case of finding a side (in Section 2), we can get a first approximation to the angle.

So, given ***A*) —, 7, 10** (Example 6 below) we select a triple that gives the smallest value when we cross-multiply and subtract:

$$\begin{array}{llll} 37^\circ) & 4 & 3 & 5 \\ A\ ) & - & 7 & 10 \end{array} \qquad \begin{array}{llll} 53^\circ) & 3 & 4 & 5 \\ A\ ) & - & 7 & 10 \end{array}$$

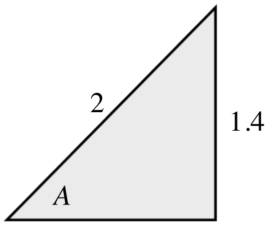
$$5\times7-3\times10=+5 \qquad 5\times7-4\times10=-5$$

Choosing the triples 4,3,5 and 3,4,5 we get the values +5 and -5, indicating that A is pretty much in the middle between 37° and 53° and so either angle could be used as a first approximation and either triple could be used as a base.

Similarly with the A) 7, —, 10 and A) 7, 10, — types we would cross-multiply and subtract in the two columns that do not contain the dash.

3.2 'Sine' Type and 'Cosine' Type

Example 6: Solve *A*) —, 1.4, 2



The process of finding a side is reversible, so that given the height and hypotenuse of a triangle we can find the angle.

In Example 1a the angle was 44° and the height was found to be 1.4. Let us reverse the process we went through: to find the angle given the height of 1.4.

This will also lead to a pattern that allows angles to be easily found.

37°	3	4	5
7°	57	7	57
44°	–	$3\times57+4\times7$	5×57
44°	–	h	2

37°	4	3	5
$x°$	57	x	57
$37°+x$	–	p	5×57
A	–	1.4	2

Above left we see the calculation as before, and on the right the set-up for the new problem.

This time we require x when h = 1.4.

Note that we can find the value of p in two ways:
1) by triple addition in the first two rows
2) by proportion in the bottom two rows.

Equating these we get 3×57 + 4x = (1.4×5×57)/2

This gives $x = \frac{5\times1.4-3\times2}{4\times2}\times 57$ and so $A = 37 + x = 37 + \frac{5\times1.4-3\times2}{4\times2}\times 57$.

This gives us the pattern we seek:

37°	**4**	**3**	**5**
A	**—**	**1.4**	**2**

That is, we cross-multiply and subtract in the last two columns for the numerator of the fraction, and we multiply as indicated by the dotted line for the denominator.

That fraction is multiplied by 57 and added to the base angle, in this case 37°.

(This result can also be arrived at through the diagram shown in Example 1a.)

We get A = 44°. [e = 0.7%]

Note that the fraction of 57 that is evaluated may be easily found since the product is only required to 1 significant figure and the numerator is always small.

Example 7: Solve A) 7, —, **9**

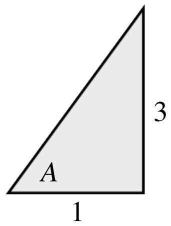
Here we can find the angle in the complementary triple and take the result from 90°.

So for 90°− A) —, 7, 9 we select 53°) 3, 4, 5 as our base triple since this minimises the difference of the cross-products: $5\times7 - 4\times9 =$ **-1**

53°	**3**	**4**	**5**
90°− A	—	**7**	**9**

We get $90° - A = 53 + \frac{5\times7-4\times9}{3\times9} \times 57 = 53 + \frac{-1}{27} \times 57 = 51°$
and so $A = 39°$. [$e = 0.4\%$]

3.3 'Tangent' Type



Example 8: Solve A) 1, 3, —

The angle is clearly over 45° so we select 53°) 3, 4, 5.

We need to find the difference of these triples:

53°	3	4	5
A	1	3	–
53°− A	$3\times1+4\times3$	$4\times1-3\times3$	–

Since this is a small triple we can equate it with "57, angle, 57":

$3\times1+4\times3$	$4\times1-3\times3$	—
57	53– A	57

These similar triangles give: $53 - A = \frac{4\times1-3\times3}{3\times1+4\times3} \times 57$
And so $A = 53 + \frac{3\times3-4\times1}{3\times1+4\times3} \times 57$, and this gives a similar pattern to the one we had in Example 4 (see Figure 4) .
We get $A = 72°$. [$e = 0.6\%$]

53°	**3**	**4**	**5**
A	**1**	**3**	—

Figure 4

Note the accuracy achieved here, even though the chosen 3,4,5 triple is 19° from the true angle.

4 Concluding Remarks

It will be seen that the patterns shown here are quite simple to use and easy to explain. Various observations that make it easier to decide which pattern to apply, and how, are available though not discussed here. A full text suitable for children in grade 8 or 9 (when Pythagoras′ Theorem is taught) is contained in four new chapters due to be added to the book "A Trillion Triangles"[2]. This new edition, which makes use of five basic/known triples will be available in January 2018 and this gives answers to at least 3 s.f. accuracy.

The complementary triples 37°)4,3,5 and 53°)3,4,5 are particularly useful: they are memorable, end in 5 which is convenient for our multiplications and divisions, and the angles are conveniently close to integral values. It is quite surprising how accurate results can be, even when only these two triples are used.

As mentioned before, any triple can be used, so we can greatly increase accuracy by making use of triples like 23°)12,5,13 or 16°)24,7,25 which have small integers and neatly fill the gap from 10° to 25° if we prefer to keep our small triples in the 0° to 10° range.

It is also worth noting that we can avoid the use of the confusing words or part-words like ′sin′ and ′inverse cosine′ etc. (though it is understood that these may be useful at some point in a student′s education).

Note also that since all plane triangles can be solved with right-angled triangles the above methods can be used on all triangles whether right-angled or not.

It was mentioned earlier that accuracy can be improved as far as required. This would be done by:
a) using a more accurate value for the known triple angle,
b) using a more accurate value for 180/π,
c) selecting a known triple close to the given one.
Given the widespread availability of electronic calculating devices however there does not seem to be any point in teaching the calculation of very accurate values.

Sides and angles of right-angled triangles can be found with ease using the four patterns shown here and would be a useful stepping-stone to more advanced work in trigonometry. And there is a lot of scope for creative work as there is a variety of ways to solve any particular triangle.

In addition to the solution of right-angled triangles there are many other applications of Pythagorean triples: in coordinate geometry, transformations, solution of trigonometric equations, proofs etc. as shown in [2], and other applications [3] in projectile motion, Simple Harmonic Motion, complex number work etc. as well as applications of higher dimensional ′triples′.

References

[1] Bharati Krishna Tirthaji Maharaja, (1994). Vedic Mathematics. Delhi: Motilal Banarasidas,.
[2] Williams. K. R. (2017). A Trillion Triangles. U.K.: Inspiration Books.
[3] Williams. K. R. (2010). Triples. U.K.: Inspiration Books.