A DEEPER LOOK INTO TIRTHAJI'S METHODS FOR GENERATING RECURRING DECIMAL STRINGS

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Introduction

In Chapter 28 of his book "Vedic Mathematics"[1] Sri Tirthaji introduces several surprisingly efficient methods for generating the recurring decimal string of a vulgar fraction. To quote Tirthaji: *"There are certain Vedic processes… by which, with the aid of what we call sahāyaks (auxiliary) fractions, the burden of subsequent operations is also considerably lightened, and the work is splendidly facilitated."* These methods are based on the fact that recurring decimal strings are generated from infinite geometric series. They employ applications of the sutras Ekadhikena Purvena (**b**y one more than the one before), Ekanyunena Purvena (**b**y one less than the one before), Nikhilam Navatascaraman Dasatah (All from 9 and the last from 10), Anurupyena (Proportionately) as well as Paravartya Yojayet (Transpose and apply). This paper explores and analyses the "Why?" behind the workings of these sutras as they are used in the generation of decimal strings.

The Conventional Approach

Tirthaji introduces two types of auxiliary fractions, ***Type 1*** and ***Type 2***. In order to investigate them, it is important to first understand the conventional algorithm used to generate a recurring decimal string. This approach involves ***prefixing the remainder in each step to a series of zeros.*** For instance, to obtain the decimal string for 1/7, because 7 cannot divide into 1, the remainder 1 is prefixed to zero (i.e. multiplied by 10) to obtain 10. When 10 is divided by 7, the quotient is 1 and the remainder is 3. The 3 is again prefixed to zero to obtain 30. When 30 is divided by 7, the quotient is 4 and the remainder is 2. The 2 is prefixed to zero to obtain 20. When 20 is divided by 7, the quotient is 2 and the remainder is 6. This process is repeated until the numerator 1 is again obtained. The string will, at this point (after six decimal digits) start recurring.

$$\begin{array}{l} \underline{7\,)\,1.\ {}^{1}0\ {}^{3}0\ {}^{2}0\ {}^{6}0\ {}^{4}0\ {}^{5}0\quad {}^{1}0\ \ldots} \\ \quad 0.\ \ 1\ \ 4\ \ 2\ \ 8\ \ 5\ \ 7\quad\ 1\ \ldots \end{array}$$

$$\begin{array}{llcl}
\text{Step 0:} & 1/7 & = & 0 + \mathbf{1}/7 \\
\text{Step 1:} & \mathbf{1}0/7 & = & 1 + \mathbf{3}/7 \\
\text{Step 2:} & \mathbf{3}0/7 & = & 4 + \mathbf{2}/7 \\
\text{Step 3:} & \mathbf{2}0/7 & = & 2 + \mathbf{6}/7 \\
\text{Step 4:} & \mathbf{6}0/7 & = & 8 + \mathbf{4}/7 \\
\text{Step 5:} & \mathbf{4}0/7 & = & 5 + \mathbf{5}/7 \\
\text{Step 6:} & \mathbf{5}0/7 & = & 7 + \mathbf{1}/7
\end{array}$$

The successive quotients in each step form the decimal string for 1/7, i.e. $0.\dot{1}4285\dot{7}$.

If 1/7 is multiplied by 7/7 the fraction 7/49 will, of course, yield the same decimal string:

$$\underline{49)\ 7.\ {}^{7}0\ {}^{21}0\ {}^{14}0\ {}^{42}0\ {}^{28}0\ {}^{35}0\quad {}^{7}0\ \ldots}$$
$$0.\ \ 1\ \ 4\ \ 2\ \ 8\ \ 5\ \ 7\ \ 1\ \ldots$$

The steps are then:

$$\begin{array}{lrcl}
\text{Step 0:} & 7/49 & = & 0 + \mathbf{7}/49 \\
\text{Step 1:} & \mathbf{7}0/49 & = & 1 + \mathbf{21}/49 \\
\text{Step 2:} & \mathbf{21}0/49 & = & 4 + \mathbf{14}/49 \\
\text{Step 3:} & \mathbf{14}0/49 & = & 2 + \mathbf{42}/49 \\
\text{Step 4:} & \mathbf{42}0/49 & = & 8 + \mathbf{28}/49 \\
\text{Step 5:} & \mathbf{28}0/49 & = & 5 + \mathbf{35}/49 \\
\text{Step 6:} & \mathbf{35}0/49 & = & 7 + \mathbf{7}/49
\end{array}$$

When each of the equations in steps 1 to 6 is multiplied by the divisor 49, the set becomes:

$$\left.\begin{array}{lrcl}
\text{Step 1:} & \mathbf{7}0 & = & 1(49) + \mathbf{21} \\
\text{Step 2:} & \mathbf{21}0 & = & 4(49) + \mathbf{14} \\
\text{Step 3:} & \mathbf{14}0 & = & 2(49) + \mathbf{42} \\
\text{Step 4:} & \mathbf{42}0 & = & 8(49) + \mathbf{28} \\
\text{Step 5:} & \mathbf{28}0 & = & 5(49) + \mathbf{35} \\
\text{Step 6:} & \mathbf{35}0 & = & 7(49) + \mathbf{7}
\end{array}\right\}\ \text{Set A}$$

These equations will be used to help explain the generation of ***Type 1*** auxillary fractions.

Type 1 Auxiliary Fractions

Using this approach, the decimal string is generated by ***prefixing the remainder in each successive step to its quotient digit***. Furthermore, the ***Type 1*** method makes use of the sutra Ekadhikena Purvena (by one more than the one before): this requires that the denominator of the fraction firstly be made to end on the digit 9 (if this is not already the case). For 1/7, the fraction is thus adapted to become 7/49 by multiplying with 7/7. Thus:

$$\frac{1}{7} = \frac{1}{7} \times \frac{7}{7}$$

We then proceed by adding "one more to the one before (the 9)" thus obtaining 4 + 1 = 5. This 5 is called the "ekadhika". Then, instead of attempting to divide 7 by 49, we start off by dividing 7 by 50 (or, effectively, 7 by 5):

$$5\,)\,7 \qquad 0.\ {}^{2}1\ {}^{1}4\ {}^{4}2\ {}^{2}8\ {}^{3}5\ {}^{0}7\quad {}^{2}1\ {}^{1}4\ \ldots$$

When 5 divides 7, the quotient is 1 and the remainder is 2. Prefixing the remainder to the quotient, we obtain 21. When 5 divides 21, the quotient is 4 and the remainder is 1. Prefixing the remainder to the quotient, we obtain 14. When 5 divides 14, the quotient is 2 and the remainder is 4. Prefixing the remainder to the quotient, we obtain 42. Proceeding thus, after six steps, the full cyclic string 142857 is once again obtained.

The steps are:

Step 1:	7/5	=	**1**	+	**2**/5		
Step 2:	**21**/5	=	**4**	+	**1**/5		
Step 3:	**14**/5	=	**2**	+	**4**/5		
Step 4:	**42**/5	=	**8**	+	**2**/5		
Step 5:	**28**/5	=	**5**	+	**3**/5		
Step 6:	**35**/5	=	**7**	+	**0**/5		

Why this algorithm works, becomes clear when each of the above six equations is multiplied by 50 (i.e. 10 × 5, where 5 is the ekadhika). We can then manipulate these equations to become identical to those in Set A, which were generated by the conventional process:

Step 1:	70	=	1(50) + 20	=	1(49+1) + 20	=	1(49) + 1 + 20	=	1(49) + 21
Step 2:	210	=	4(50) + 10	=	4(49+1) + 10	=	4(49) + 4 + 10	=	4(49) + 14
Step 3:	140	=	2(50) + 40	=	2(49+1) + 40	=	2(49) + 2 + 40	=	2(49) + 42
Step 4:	420	=	8(50) + 20	=	8(49+1) + 20	=	8(49) + 8 + 20	=	8(49) + 28
Step 5:	280	=	5(50) + 30	=	5(49+1) + 30	=	5(49) + 5 + 30	=	5(49) + 35
Step 6:	350	=	7(50) + 0	=	7(49+1) + 0	=	7(49) + 7 + 0	=	7(49) + 7

The *Type 1* auxiliary fraction method starts with dividing 7 by the ekadhika 5 (or 70 by 50), though 70 must, of course, actually be divided by 49, as 70/49 = 10 × 1/7. Only then will the correct decimal string for 1/7 be obtained. Each step of the Ekadhika division process thus employs a divisor which is effectively *greater* than it should be. The remainder obtained in each successive division step is thus *smaller* than it should be. So, an additional amount must be added to each successive remainder, before it can serve as the new dividend. This amount turns out to be the quotient itself! Because there is a difference of 1 between 50 and 49, the above equations show that the quotient must effectively be multiplied by 1 before being added to the remainder. Then the correct dividend for the next step is revealed. For instance, in Step 3 above, (1×2) is added to the remainder 40 to obtain 42. Then 42 divided by 5 (or 420 divided by 50) produces the next quotient. This is why, in the *Type 1* method, the remainder is prefixed to the quotient and not to a zero, as is done when employing the conventional technique.

Depending on the nature of a particular fraction, it is often easier to find the recurring string using the *Type 1* approach. For instance, the respective strings for 1/19, 1/29 and 1/59 can be generated using divisions by 2, 3 and 6 respectively, instead of by 19, 29 and 59.

$$\begin{array}{ll} \text{Conventional approach:} & \underline{0.\,0\ 5\ 2\ 6\ 3\ 1\ 5\ 7\ 8\ 9\ 4\ 7\ 3\ 6\ 8\ 4\ 2\ 1\quad 0\ 5\ 2\ 6\ldots} \\ & 19\,/\,1.{}^{1}0{}^{10}0{}^{5}0{}^{12}0{}^{6}0{}^{3}0{}^{11}0{}^{15}0{}^{17}0{}^{18}0{}^{9}0{}^{14}0{}^{7}0{}^{13}0{}^{16}0{}^{8}0{}^{4}0{}^{2}0\quad {}^{1}0{}^{10}0{}^{5}0{}^{12}0\ldots \\ \textbf{\textit{Type 1}}\text{ approach:} & 2\,)\,1\quad 0.\,{}^{1}0\ 5\ {}^{1}2\ 6\ 3\ {}^{1}1\ {}^{1}5\ {}^{1}7\ {}^{1}8\ 9\ {}^{1}4\ 7\ {}^{1}3\ {}^{1}6\ 8\ 4\ 2\ 1\ldots \end{array}$$

Fractions With Denominators More Than One Unit Below A Multiple of 10

The previous section addresses cases which have denominators ***one*** unit below a multiple of 10. The sutra Ekadhikena Purvena can also be applied to fractions with denominators *further* *than one unit* below such a multiple. In such cases, each successive remainder is prefixed to a *multiple* of the quotient for that particular step. The factor by which the quotient is multiplied equals the number of units the denominator happens to be below the nearest multiple of 10. Thus, in finding the decimal string for, say, 150/68 (where the denominator 68 is ***two*** units below 70), we proceed as follows:

$$\frac{150}{68} = \frac{15}{6.8} \rightarrow \frac{15}{7}$$

Here the "one before (the 8)" is 6. One more than 6 is 7. The "ekadhika" is thus 7. We then proceed to divide by 7 instead of 6.8, correcting each successive new quotient as we proceed:

$${}^{1}2^{0}2^{4}0^{5}5^{4}8^{0}8^{2}2^{3}3^{1}5^{6}2^{1}9^{0}4^{1}1\ldots$$

$$\times2\downarrow\ \times2\downarrow\ \times2\downarrow\ \times2\downarrow\ \times2\downarrow\ \times2\downarrow\ \times2\downarrow\ \times2\downarrow\ \times2\downarrow\ \times2\downarrow\ \times2\downarrow\ \times2\downarrow\ \times2\downarrow$$

$${}^{1}4^{0}4^{4}0^{6}0^{5}6^{1}6^{2}4^{3}6^{2}0^{6}4^{2}8^{0}8^{1}2\ldots$$

Division of 15 by 7 yields 2 remainder 1. The quotient 2 is multiplied by 2 to become 4 and the remainder 1 is prefixed to the 4 to become 14 (effectively 1×10 is added to 4). Division of 14 by 7 yields 2 remainder 0. The quotient 2 is multiplied by 2 to become 4 and the remainder 0 is prefixed to the 4 to remain 4. Division of 4 by 7 yields 0 remainder 4. The quotient 0 is multiplied by 2 to remain 0 and the remainder 4 is prefixed to the 0 to become 40 (effectively 4×10 is added to 0). Division of 40 by 7 yields 5 remainder 5. The quotient 5 is multiplied by 2 to become 10 and the remainder 5 is prefixed to the 10 to become 60. This is so because prefixing a remainder of 5 to 10 really means adding 5×10 to 10. The 60 is then divided by 7 to yield 8 remainder 4… and so the decimal 2.2058823532941 ... etc. is generated.

The diagram below illustrates why this algorithm works.

$\frac{150}{70}$ $\begin{array}{r} 2 \\ 70\,\overline{)\,1\,5\,0} \\ \underline{1\,4\,0} \\ 1\,0 \end{array}$ $\frac{150}{70} = 2 + \frac{10}{70}$ $150 = 2(70) + 10 = 2(68 + 2) + 10 = 2(68) + 4 + 10$	$\frac{150}{68}$ $\begin{array}{r} 2 \\ 68\,\overline{)\,1\,5\,0} \\ \underline{1\,3\,6} \\ 1\,4 \end{array}$ $\frac{150}{68} = 2 + \frac{14}{68}$ $150 = 2(68) + 14$
$\frac{140}{70} = 2 + \frac{0}{70}$ $140 = 2(70) + 0 = 2(68 + 2) + 0 = 2(68) + 4 + 0$	$\frac{140}{68} = 2 + \frac{4}{68}$ $140 = 2(68) + 4$
$\frac{40}{70} = 0 + \frac{40}{70}$ $40 = 0(70) + 40 = 0(68 + 2) + 40 = 0(68) + 0 + 40$	$\frac{40}{68} = 0 + \frac{40}{68}$ $40 = 0(68) + 40$
$\frac{400}{70} = 5 + \frac{50}{70}$ $400 = 5(70) + 50 = 5(68 + 2) + 50 = 5(68) + 10 + 50$	$\frac{400}{68} = 5 + \frac{60}{68}$ $400 = 5(68) + 60$

The right-hand column shows the steps used by the conventional division process in finding the decimal string for 150/68 (or 15/6.8). The left-hand column depicts the steps used in the process of 150 being divided by ***70*** (or, proportionally, 15 divided by the ekadhika 7), and how each step must necessarily be adapted to be identical to the steps in the right-hand column. Because the divisor 70 is greater than the divisor 68 (with a difference of 2), the new dividend must equal the remainder plus ***two*** times the quotient. This answer is then identical to the remainder in the corresponding step of the conventional process. Division of 7 into this new dividend (or 70 into ten times this value) produces the quotient in the next step.

Applying this method to, say, 4/17 - where the denominator is ***3 units*** below 20 - each remainder must be prefixed to ***3 times*** the quotient in each step. In this instance, many of the successive dividends become very large, so vinculum numbers can be employed as remainders. The ekadhika here is 2:

$$\frac{4}{17} = 0.\ \begin{array}{ccccccc} 2 & 3 & {}^{\bar{1}}5 & {}^{1}2 & {}^{\bar{2}}9 & {}^{\bar{1}}4 & 1\ldots \\ {\scriptstyle\times3}\downarrow & {\scriptstyle\times3}\downarrow & {\scriptstyle\times3}\downarrow & {\scriptstyle\times3}\downarrow & {\scriptstyle\times3}\downarrow & {\scriptstyle\times3}\downarrow & {\scriptstyle\times3}\downarrow \\ 6 & 9 & 5 & {}^{1}6 & 7 & 2 & 3\ldots \end{array}$$

Infinite Geometric Series and the Ekadhika

Returning to the example of 1/7 (or 7/49), the ***Type 1*** approach effectively employs 50 (i.e. ten times the ekadhika) as the divisor of 7, rather than 49. As we have seen in the previous section, in order to obtain the correct decimal string for 7/49 if the divisor in each step happens to be ***one more than*** 49, a small amount (let it be *x*) must be added to each successive remainder to obtain the correct dividend for the next step. We can make use of this fact to generate the infinite geometric series for 1/7. An iterative procedure is followed:

$$\frac{7}{49} = \frac{7}{50} + x \qquad \text{But } x = \frac{7}{49} - \frac{7}{50} = 7\left(\frac{1}{49} - \frac{1}{50}\right) = 7\left(\frac{1}{49\times50}\right) = \frac{1}{50}\left(\frac{7}{49}\right)$$

Thus $\frac{7}{49} = \frac{7}{50} + \frac{1}{50}\left(\frac{7}{49}\right)$ But $\frac{7}{49} = \frac{7}{50} + \frac{1}{50}\left(\frac{7}{49}\right)$

Thus $\frac{7}{49} = \frac{7}{50} + \frac{1}{50}\left(\frac{7}{50} + \frac{1}{50}\left(\frac{7}{49}\right)\right)$ But $\frac{7}{49} = \frac{7}{50} + \frac{1}{50}\left(\frac{7}{49}\right)$

Thus $\frac{7}{49} = \frac{7}{50} + \frac{1}{50}\left(\frac{7}{50} + \frac{1}{50}\left(\frac{7}{50} + \frac{1}{50}\left(\frac{7}{49}\right)\right)\right)$ etc.

Thus $\frac{7}{49} = \frac{7}{50} + \frac{7}{(50)^2} + \frac{7}{(50)^3} + \frac{7}{(50)^4} + \frac{7}{(50)^5} + \cdots$

Thus $\frac{1}{7} = 7\times(\frac{1}{50} + \frac{1}{(50)^2} + \frac{1}{(50)^3} + \frac{1}{(50)^4} + \frac{1}{(50)^5} + \cdots)$

and $\frac{1}{7} = 7\times(0.02 + 0.0004 + 0.000008 + 0.00000016 + \cdots)$

so $\frac{1}{7} = 7\times(0.0204081632\ldots) = 0.142857\ldots$

Thus also $\frac{1}{7} = 7\times\sum_{n=1}^{\infty}\left(\frac{1}{50}\right)^n = 7\times\sum_{n=1}^{\infty}\left(\frac{1}{49+1}\right)^n$

The fact that 1/7 is a multiple of the decimal string for 1/49 = 0.0204081632… - which itself consists of a geometric series with common ratio 0.02 - also demonstrates why the string for 1/7 appears to be made up of multiples of 7, i.e. 14, 28 and 56 (+1 due to the carry over when 7 is multiplied by 16).

The Use of 49, 499, 4999, 49999 etc.

We have seen that $\frac{1}{49} = \sum_{n=1}^{\infty}\left(\frac{1}{50}\right)^n = 0.020408163264 \ldots$

This fact can be very useful in generating the decimal string for any multiple of this fraction,

e.g. $\frac{3}{49} = 3\times(0.0204081632 \ldots) = 0.0612244896 \ldots$

Similarly, $\frac{1}{499} = \sum_{n=1}^{\infty}\left(\frac{1}{500}\right)^n = 0.002004008016032064 \ldots$

Thus $\frac{12}{499} = 12\times(0.0204081632 \ldots) = 0.024048096192384768 \ldots$

The fact that, for instance, 499999 is not a prime number, as 499999 = $31\times(127)^2$, can facilitate finding the decimal of, for instance, $\frac{1}{127}$:

$$\frac{1}{127} = \frac{1}{127}\times\frac{127\times31}{127\times31} = \frac{3937}{499999} = 3937\times\sum_{n=1}^{\infty}\left(\frac{1}{500000}\right)^n$$

$$= 3937\times 0.000002000004000008\ldots = 0.007874015748031496\ldots$$

The use of 9, 99, 999, 9999 etc.

For cases where only 9s appear in a denominator, finding the decimal string is even easier:

$\frac{1}{9} = \sum_{n=1}^{\infty}\left(\frac{1}{10}\right)^n = 0.111111111 \ldots$ and $\frac{1}{99} = \sum_{n=1}^{\infty}\left(\frac{1}{100}\right)^n = 0.010101010101 \ldots$

and $\frac{1}{999} = \sum_{n=1}^{\infty}\left(\frac{1}{1000}\right)^n = 0.001001001001001 \ldots$ etc.

Thus, for instance, $\frac{673}{999} = 673\times\sum_{n=1}^{\infty}\left(\frac{1}{1000}\right)^n = 0.673673673673 \ldots$

Also, for instance, $\frac{4}{33} = \frac{4}{33}\times\frac{3}{3} = \frac{12}{99} = 12\times\sum_{n=1}^{\infty}\left(\frac{1}{100}\right)^n = 0.12121212 \ldots$

and $\frac{43}{666} = \frac{43}{666}\times\frac{1.5}{1.5} = \frac{65.5}{999} = 65.5\sum_{n=1}^{\infty}\left(\frac{1}{100}\right)^n = 0.0655655655 \ldots$

The last two examples illustrate how the sutra Anurupyena (proportionately) can be employed in the generation of decimal strings.

Indeed, any number 1/*N* (which has a perfectly recurring decimal string) can be multiplied by some ratio *x/x* to convert the denominator to a string of 9s: the number of 9s making up this denominator must equal the length of one cycle of the recurring decimal string related to $1/N$. The digits in x will then be identical to the digits in one full cycle of the string.

For instance, the decimal string for 1/7 can be found using this approach:

$$\frac{1}{7} = \frac{1}{7} \times \frac{x}{x} = \frac{x}{999999}$$

Solving for x yields: $x = \frac{999999}{7}$

$$\begin{array}{l} 7\overline{)9^2 9^1 9^5 9^3 9^4 9} \\ \quad 1\ 4\ 2\ 8\ 5\ 7\ rem\ 0 \quad = x \end{array}$$

This example is relevant to the discussion on the working of ***Type 2*** auxiliary fractions.

Type 2 **Auxiliary Fractions**

The ***Type 1*** method can be used on fractions with denominators ending on the digit 9, i.e. denominators that are one unit ***below*** a multiple of 10. It can be widely used; for instance, fractions with denominators ending respectively on 1, 3 and 7 can easily be adapted by multiplying both the numerator and the denominator by 9, 3 and 7 respectively.

e.g. $\frac{2}{13} = \frac{2\times3}{13\times3} = \frac{6}{69}$ (Here the ekadhika is 7) and $\frac{1}{21} = \frac{1\times9}{21\times9} = \frac{9}{189}$ (Here the ekadhika is 19)

The very last example demonstrates an instance where using a ***Type 1*** auxiliary fraction does ***not*** facilitate the generation of the decimal string. For the fraction $\frac{1}{21}$ - with ekadhika 19 - the process of dividing 9 by 19 is equally (if not more) laborious compared to dividing 1 by 21. This is where the ***Type 2*** auxiliary fraction comes into play. To quote Tirthaji: "*If a fraction has a denominator ending on 1, drop the 1 and decrease the numerator by unity*." We will now proceed to show how finding the decimal string in this way, involves a process of attempting to turn the *denominator* into a string of 9's, in turn causing the *numerator* to become exactly one period of the cyclic string!

Tirthaji's use of the ***Type 2*** auxiliary fraction approach to help find the decimal string for $\frac{1}{21}$ is discussed below. It is then followed by an analysis of why the algorithm works.

"Drop the 1 and decrease the numerator by unity."

$$\frac{1}{21} \xrightarrow{-1} \frac{0}{2}$$

Both processes of dropping the 1 as well as decreasing the numerator by unity can be seen as applications of the sutra Ekanyunena Purvena (by one ***less*** than the one before).

The next step is to subtract the new numerator 0 from 9**.** When this answer 9 (which is the complement of 0 from 9) is divided by the new denominator 2, the answer is 4 remainder 1.

$$\begin{array}{l} 0.\ {}^{0}0\ \ {}^{1}4\ \ {}^{1}7\ \ {}^{0}6\ \ {}^{1}1\ \ {}^{0}9\quad {}^{0}0\ {}^{1}4\ \ldots \\ \quad\ 9\ \ {}^{1}5\ \ {}^{1}2\ \ {}^{0}3\ \ {}^{1}8\ \ {}^{0}0\quad {}^{0}9\ {}^{1}5\ \ldots \end{array}$$

The complement of 4 from 9 is 5. Prefixing the remainder 1 to the 5 yields 15, which becomes the new dividend. 15 divided by 2 yields 7 remainder 1. The complement of 7 from 9 is 2. Prefixing the remainder 1 to the 2 yields 12, which becomes the new dividend. 12 divided by 2 yields 6 remainder 0. Continuing thus, the full six-digit cyclic string $0.\dot{0}4761\dot{9}$ is revealed.

$$\frac{1}{21} = \begin{array}{l} 0.{}^{0}0\ {}^{1}4\ {}^{1}7\ {}^{0}6\ {}^{1}1\ {}^{0}9\ {}^{0}0\ {}^{1}4 \\ \div 2 \\ \quad 9\ {}^{1}5\ {}^{1}2\ {}^{0}3\ {}^{1}8\ {}^{0}0\ {}^{0}9 \end{array}$$

To summarise the process: when each quotient in the first row is taken away from 9, that answer (with the remainder prefixed to it) becomes the new dividend. This is an application of the sutra Nikhilam Navatascaraman Dasatah (all from 9 and the last from 10). As the cycle infinitely repeats, "the last" can never be subtracted from 10.

In order to explain why this process works, we will start by multiplying the fraction $\frac{1}{21}$ by $\frac{x}{x}$:

$$\frac{1}{21} = \frac{1}{21} \times \frac{x}{x} = \frac{x}{999999}$$

where x is one full cycle of the 6-digit recurring string.

Solving for x yields: $$x = \frac{999999}{21}$$

$$\begin{array}{l} \underline{21\,)\ 9\ {}^{9}9\ {}^{15}9\ {}^{12}9\ {}^{3}9\ {}^{18}9\ \ldots} \\ \qquad\ \ 0\ \ 4\ \ \ 7\ \ \ 6\ \ \ 1\ \ \ 9\ \ldots \end{array}$$

The steps in this conventional division are:

Step 1:	9/21	=	0 + 9/21
Step 2:	99/21	=	4 + 15/21
Step 3:	159/21	=	7 + 12/21
Step 4:	129/21	=	6 + 3/21
Step 5:	39/21	=	1 + 18/21
Step 6:	189/21	=	9 + 0/21

When each of the equations in steps 1 to 6 is multiplied by the divisor 21, the set becomes:

$$\left.\begin{array}{lrcl} \text{Step 1:} & 9 & = & 0(21) + 9 \\ \text{Step 2:} & 99 & = & 4(21) + 15 \\ \text{Step 3:} & 159 & = & 7(21) + 12 \\ \text{Step 4:} & 129 & = & 6(21) + 3 \\ \text{Step 5:} & 39 & = & 1(21) + 18 \\ \text{Step 6:} & 189 & = & 9(21) + 0 \end{array}\right\} \text{Set B}$$

However, the six steps used by the *Type 2* string generation method employ a divisor of 2 and not 21. They are shown below. Note that the addition of the number 10 in steps 3, 4 and 6 is analogous to prefixing the remainder to a complement from 9.

$$\begin{array}{lrcl c rcl}
\text{Step 1:} & 0/2 & = & 0 + 0/2 & \Longrightarrow & 0/2 & = & 0 + 0/2 \\
\text{Step 2:} & (9-0)/2 & = & 4 + \mathbf{1}/2 & \Longrightarrow & 9/2 & = & 4 + \mathbf{1}/2 \\
\text{Step 3:} & (\mathbf{10} + 9-4)/2 & = & 7 + \mathbf{1}/2 & \Longrightarrow & 15/2 & = & 7 + \mathbf{1}/2 \\
\text{Step 4:} & (\mathbf{10} + 9-7)/2 & = & 6 + 0/2 & \Longrightarrow & 12/2 & = & 6 + 0/2 \\
\text{Step 5:} & (9-6)/2 & = & 1 + \mathbf{1}/2 & \Longrightarrow & 3/2 & = & 1 + \mathbf{1}/2 \\
\text{Step 6:} & (\mathbf{10} + 9-1)/2 & = & 9 + 0/2 & \Longrightarrow & 18/2 & = & 9 + 0/2
\end{array}$$

This set of equations can be manipulated to become:

$$\begin{array}{lrcl c rcl c l c l}
\text{Step 1:} & 0 & = & 0(2) + 0 & \text{or} & 0 & = & 0(20) + 0 & = & 0(21-1) + 0 & = & 0(21) + 0 - 0 \\
\text{Step 2:} & 9 & = & 4(2) + 1 & \text{or} & 90 & = & 4(20) + 10 & = & 4(21-1) + 10 & = & 4(21) + 10 - 4 \\
\text{Step 3:} & 15 & = & 7(2) + 1 & \text{or} & 150 & = & 7(20) + 10 & = & 7(21-1) + 10 & = & 7(21) + 10 - 7 \\
\text{Step 4:} & 12 & = & 6(2) + 0 & \text{or} & 120 & = & 6(20) + 0 & = & 6(21-1) + 0 & = & 6(21) + 0 - 6 \\
\text{Step 5:} & 3 & = & 1(2) + 1 & \text{or} & 30 & = & 1(20) + 10 & = & 1(21-1) + 10 & = & 1(21) + 10 - 1 \\
\text{Step 6:} & 18 & = & 9(2) + 0 & \text{or} & 180 & = & 9(20) + 0 & = & 9(21-1) + 0 & = & 9(21) + 0 - 9
\end{array}$$

Each step in the *Type 2* auxiliary fraction method involves division by the ekadhika 2 (or, proportionately, by 20), though the division should actually be done by 21. Only then will the correct decimal string for 1/21 be obtained. Each step of the *Type 2* division process thus employs a divisor which is effectively *smaller* than it should be. The remainder obtained in each successive division step is thus *greater* than it should be. So, in the process of finding the next dividend, an additional amount must be subtracted from each successive remainder. This amount turns out to be the quotient itself! Because there is a difference of 1 between 21 and 20, the above equations show that the quotient must effectively be multiplied by 1 before being subtracted from the remainder. It is also evident that the answer obtained thus, still does not yield the correct dividend.

However, (as shown below) if 9 is added to both sides of each equation, and we then rather subtract the quotient from the 9 (instead of from the remainder) and then only add the remainder, then this process yields the correct dividend (i.e. it is identical to the dividend used in each step of the conventional process – refer to Set B). For instance, in Step 3 below, (1×7) is subtracted from 9 to obtain 2. When the remainder 10 is then added (i.e. 1 is prefixed to the 2) the correct dividend for the next step, i.e. 12 is found. This is why, in the ***Type 2*** method, the remainder is prefixed to the *complement* of the quotient from 9 and not to the quotient itself.

$$\begin{array}{lrcl c l}
\text{Step 1:} & 9 & = & 0(21) + 0 + 9 - 0 & = & 0(21) + 9 \\
\text{Step 2:} & 99 & = & 4(21) + 10 + 9 - 4 & = & 4(21) + 15 \\
\text{Step 3:} & 159 & = & 7(21) + 10 + 9 - 7 & = & 7(21) + 12 \\
\text{Step 4:} & 129 & = & 6(20) + 0 + 9 - 6 & = & 6(21) + 3 \\
\text{Step 5:} & 39 & = & 1(20) + 10 + 9 - 1 & = & 1(21) + 18 \\
\text{Step 6:} & 189 & = & 9(20) + 0 + 9 - 9 & = & 9(21) + 0
\end{array}$$

While 1/21 has a denominator that is ***one*** unit more than a multiple of 10, it should also be possible (by multiplying the quotient with a factor before finding its complement) to use the ***Type 2*** method on fractions with denominators *further than one unit* above such a base.

Conclusion

An investigation into the nature of decimal strings and the patterns contained in them, is greatly facilitated by an understanding of the methods by which they are generated. In Chapter 28 of his book "Vedic Mathematics", Sri Tirthaji displays a deep understanding of division and the geometric series which underly all infinitely recurring decimal strings. He brilliantly and creatively harnesses many of the Vedic sutras to help produce these strings far more quickly and efficiently than conventionally used methods.

This paper has analysed only some of the techniques explained in Chapter 28. There is scope for further work on this subject. The Ekadhikena Purvena sutra (used in the ***Type 1*** auxiliary fraction method) has been succsssfully used to generate binary strings in the investigation of prime numbers[2,3]. It is possible that some of the other Vedic string generation techniques could be employed in a similar way.

Quote by Sri Tirthaji at the end of Chapter 28: "In fact, the very discovery of these auxiliaries and of their wonderful utility in the transmogrification of frightful looking denominators of vulgar fractions into such simple and easy denominator-divisors must suffice to prepare the scientifically-minded seeker after knowledge for the marvellous devices still further on in the offing."

References

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