FINDING SUMS OF POWERS OF ROOTS OF POLYNOMIALS

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Abstract

This paper shows how we can use a result from Bharati Krishna Tirthaji together with the *Transpose and Apply* division process of Vedic Mathematics to easily get sums of any like powers of roots of any polynomial without the need to actually find the roots themselves.

1. Introductory

In Chapter 22 of Tirthaji's book[1] he looks at the relationship between the factors of a polynomial and its successive differentials.

The first three examples he gives by way of illustration are as follows:

(1) $x^2 + 3x + 2 = \overset{a}{(x + 1)}\overset{b}{(x + 2)}$

$\therefore D_1$ (the first differential) $= 2x + 3 = (x + 1) + (x + 2) = \Sigma a$

(2) $x^3 + 6x^2 + 11x + 6 = \overset{a}{(x + 1)}\overset{b}{(x + 2)}\overset{c}{(x + 3)}$

$\therefore D_1 = 3x^2 + 12x + 11 = (x^2 + 3x + 2) + (x^2 + 5x + 6) + (x^2 + 4x + 3)$

$= ab + bc + ac = \Sigma ab.$

$D_2 = 6x + 12 = 2(3x + 6) = 2[(x + 1) + (x + 2) + (x + 3)]$

$= 2(a + b + c) = 2\Sigma a = 2!\,\Sigma a.$

(3) $x^4 + 10x^3 + 35x^2 + 50x + 24 = (x + 1)(x + 2)(x + 3)(x + 4)$

$\therefore D_1 = 4x^3 + 30x^2 + 70x + 50 = \Sigma abc$

$D_2 = 12x^2 + 60x + 70 = 2\Sigma ab = 2!\Sigma ab$

$D_3 = 24x + 60 = 6(4x + 10) = 3!\Sigma a.$

Our interest here is only in the first differential of a polynomial of any degree (the degree of the polynomials in the above examples are 2, 3 and 4 respectively).

So we can say that the first differential of a polynomial of degree n in which the leading coefficient is 1, is the *sum of the products* of its factors taken $n - 1$ at a time. (1)

This simple result can be surprisingly effective in finding sums of powers of roots of polynomials and is proved in Appendix 1.

The relevant Sutra is *Guṇaka-Samuccaya*: *All the Multipliers*. The 'multipliers' here would refer to the factors.

2. Application of Result (1)

We begin with quadratics as these most simply illustrate the method, which is easily extended to polynomials of higher degree.

2.1 Example 1

The quadratic $x^2 - 3x + 2$ is easily found to have roots 1 and 2.

Suppose we require the sum of the roots, the sum of the squares of the roots, the sum of the cubes of the roots etc.

We can do it like this:

Consider the fraction $\dfrac{2x-3}{x^2-3x+2}$ where the denominator is the given polynomial, and the numerator is the first derivative of the denominator (and also the sum of the factors by result (1) above).

There are two things we can do with this fraction:

A) we can divide the numerator by the denominator to get an infinite polynomial.

B) we can factorise the denominator, replace the numerator with the sum of these factors, split into two fractions, cancel and then divide and combine these.

The results of A and B will of course be equivalent.

If we look at A above and divide the expressions, using the Paravartya Sutra (*Transpose and Apply*) we have:

$$\begin{array}{l|lllll}
x^2-3x+2 & 2x & -3 & & & \\
+3 \quad -2 & & 6 & -4 & & \\
 & & & 9 & -6 & \\
 & & & & 15 & -10 \\
\hline
 & \multicolumn{5}{l}{\dfrac{2}{x}+\dfrac{3}{x^2}+\dfrac{5}{x^3}+\dfrac{9}{x^4}+\ldots}
\end{array} \qquad (2)$$

For B above we get $\dfrac{(x-1)+(x-2)}{(x-1)(x-2)}$ and splitting this into two fractions we have,

$$\frac{(x-1)}{(x-1)(x-2)}+\frac{(x-2)}{(x-1)(x-2)}=\frac{1}{(x-1)}+\frac{1}{(x-2)}.$$

Next expanding these using the Paravartya Sutra we have:

$$\frac{1}{x}+\frac{1}{x^2}+\frac{1^2}{x^3}+\frac{1^3}{x^4}+\ldots+\frac{1}{x}+\frac{2}{x^2}+\frac{2^2}{x^3}+\frac{2^3}{x^4}+\ldots$$
$$=\frac{2}{x}+\frac{1+2}{x^2}+\frac{1^2+2^2}{x^3}+\frac{1^3+2^3}{x^4}+\ldots \qquad (3)$$

We note that the numerators of the fractions in this last result are respectively, the number of roots of theroots is $1^3 + 2^3 = 9$ (the 4th numerator),
etc.

2.2 Example 2 – Irrational Roots

Taking another example, suppose that we require the sum of the cubes of the roots of $x^2 + 5x + 2$.

We simply divide $x^2 + 5x + 2$ into its derivative, $2x + 5$ and go as far as the term with x^4 in the denominator:

$$\begin{array}{l|llll}
x^2 + 5x + 2 & 2x + 5 & & & \\
\quad -5 \quad -2 & -10 & -4 & & \\
 & & 25 & 10 & \\
 & & & -105 & -42 \\
\hline
 & \frac{2}{x} - \frac{5}{x^2} & + \frac{21}{x^3} & - \frac{95}{x^4} + \ldots & \\
\end{array}$$

We see the required result is -95.

Note that this quadratic cannot be factorised, so that finding the roots and then cubing them would be a great deal of work.

2.3 Example 3 – Complex Roots

And even if the quadratic has no real roots (i.e. they are complex numbers) we can use this method.

Find the sum of the squares and sum of the cubes of the roots of $x^2 - 2x + 3$.

$$\begin{array}{l|llll}
x^2 - 2x + 3 & 2x - 2 & & & \\
\quad +2 \quad -3 & 4 & -6 & & \\
 & & 4 & -6 & \\
 & & & -4 & 6 \\
\hline
 & \frac{2}{x} + \frac{2}{x^2} & - \frac{2}{x^3} & - \frac{10}{x^4} + \ldots & \\
\end{array}$$

So the required values are -2 and -10.

Again, this would involve a colossal amount of work if done by first finding the roots themselves.

3. Simplifying Division Using *Vertically and Crosswise*

In fact the divisions shown above can be simplified further[2].

The last calculation above can look like this:

$$\begin{array}{l}
x^2 - 2x + 3 \;)\; 2x - 2 \\
\quad {\scriptstyle +2 \quad -3}\;)\underline{\hspace{6cm}} \\
\qquad\qquad \underline{\dfrac{2}{x} + \dfrac{2}{x^2} - \dfrac{2}{x^3} - \dfrac{10}{x^4} + \ldots}
\end{array}$$

That is, having got the first numerator, 2, we multiply it by the 1st transposed value (+2) to get +4 and add this in the next column: -2 + 4 = +2. This is the 2nd numerator.

$$\begin{array}{c} +2 \\ | \\ 2 \end{array}$$

To continue we cross-multiply the last two numerators (2 and 2) by the two transposed values and add:
2×+2 + 2×-3 = -2. This is the 3rd numerator.

$$\begin{array}{ccc} +2 & & -3 \\ & \times & \\ 2 & & 2 \end{array}$$

Then cross-multiplying the last two numerators (2 and -2) with the transposed digits again, and adding:
-2×+2 + 2×-3 = -10, the 4th numerator. And so on as far as needed.

$$\begin{array}{ccc} +2 & & -3 \\ & \times & \\ 2 & & -2 \end{array}$$

This is clearly an application of the *Vertically and Crosswise* Sutra.

4. A Cubic

4.1 Example 4

Let us take a cubic next. The method is just the same.

Find the sum of the squares and sum of the cubes of the roots of $x^3 - 2x^2 - 11x + 12$.

We require $\dfrac{3x^2 - 4x - 11}{x^3 - 2x^2 - 11x + 12}$, where again the numerator is the differential of the denominator.

Now, we divide:

$$\begin{array}{l}
x^3 - 2x^2 - 11x + 12 \;)\; 3x^2 - 4x - 11 \\
\quad {\scriptstyle 2 \quad\; 11 \quad\; -12}\;) \quad {\scriptstyle 6 \quad\; 33 \quad\; -36} \\
\qquad\qquad\qquad\qquad\quad {\scriptstyle 4 \quad\; 22 \quad\; -24} \\
\qquad\qquad\qquad\qquad\qquad\;\; \underline{{\scriptstyle 52 \;.\;.\;.\;.}} \\
\qquad\qquad\qquad \underline{\dfrac{3}{x} + \dfrac{2}{x^2} + \dfrac{26}{x^3} + \dfrac{38}{x^4} + \ldots}
\end{array}$$

So the required values are 26 and 38.

The actual roots here are 1, -3, 4. A factorisable equation was chosen so that the answers can be easily checked, but the roots could be irrational or complex or both.

Again, the lines of working (6   33   -36 etc.) can be avoided and are shown here to aid with understanding the method. Using only the transposed digits and the quotient coefficients as they appear the calculation would look like this:

$$\begin{array}{l} x^3 - 2x^2 - 11x + 12 \;)\; 3x^2 - 4x - 11 \\ \quad 2 \quad 11 \quad -12 \;)\underline{\hphantom{3x^2 - 4x - 11 xxxxx}} \\ \qquad\qquad\qquad \underline{\dfrac{3}{x} + \dfrac{2}{x^2} + \dfrac{26}{x^3} + \dfrac{38}{x^4} + \ldots} \end{array}$$

Initially we obtain $3x^2 \div x^3 = 3/x$, using *The First by the First*, then 3×2 (the 1st transposed digit) = 6.
Add this to the x coefficient in the dividend to get 2 as the 2nd quotient coefficient.

$$\begin{array}{cc} 2 & 11 \\ 3 & 2 \end{array}$$

Then multiply crosswise with 2, 11 and 3, 2 to get 4 + 33 = 37 and add this to the last coefficient in the dividend to get 37 – 11 = 26 as the 3rd quotient coefficient.

$$\begin{array}{ccc} 2 & 11 & -12 \\ 3 & 2 & 26 \end{array}$$

Finally we cross-multiply the three transposed values with the three quotient coefficient values found: $26\times2 + 3\times(-12) + 11\times2 = 52 - 36 + 22 = 38$ for the 4th quotient coefficient.

4.2 Proof

We may prove this result in a similar way to that for quadratics as follows.

$$\frac{3x^2-4x-11}{x^3-2x^2-11x+12} = \frac{(x-1)(x+3)+(x-1)(x-4)+(x+3)(x-4)}{(x-1)(x+3)(x-4)} \text{ (using result (1))}$$

$$= \frac{1}{(x-4)} + \frac{1}{(x+3)} + \frac{1}{(x-1)} \text{ (splitting into three fractions)}$$

Then following the process we used for quadratics we expand in two ways:

A: dividing $x^3 - 2x^2 - 11x + 12$ into its derivative we get $\dfrac{3}{x} + \dfrac{2}{x^2} + \dfrac{26}{x^3} + \dfrac{38}{x^4} + \ldots$

B: dividing the three terms $\dfrac{1}{(x-4)} + \dfrac{1}{(x+3)} + \dfrac{1}{(x-1)}$ we get

$$\frac{1}{x}+\frac{4}{x^2}+\frac{4^2}{x^3}+\frac{4^3}{x^4}+\ldots \quad \frac{1}{x}-\frac{3}{x^2}+\frac{3^2}{x^3}-\frac{3^3}{x^4}+\ldots \quad \frac{1}{x}+\frac{1}{x^2}+\frac{1^2}{x^3}+\frac{1^3}{x^4}+\ldots$$

$$= \frac{3}{x}+\frac{4+-3+1}{x^2}+\frac{4^2+(-3)^2+1^2}{x^3}+\frac{4^3+(-3)^3+1^3}{x^4}+\ldots$$

And we see that comparing the terms of like power the various terms in A are respectively the number of roots, the sum of the roots, the sum of the squares of the roots, the sum of the cubes of the roots and so on.

5. Leading Coefficient Not Unity

The leading coefficient does not need to be unity for this process to be applied as the next example shows.

5.1 Example 5

Find the sum of the roots and the sum of the squares of the roots of $2x^2 - 7x + 3$.

We require $\dfrac{4x-7}{2x^2-7x+2}$.

Such divisions are explained by Tirthaji on page 69 of his book: we can transpose as usual but due to the coefficient of x^2 being 2 instead of 1 we need to divide each coefficient obtained (after the initial one) by 2.

We get:

$$\begin{array}{l} 2x^2 - 7x + 3 \;)\; 4x - 7 \\ \quad\;\; {\scriptstyle +7 \quad -3} \;)\overline{\qquad\qquad\qquad} \\ \qquad\qquad \dfrac{2}{x} + \dfrac{3\frac{1}{2}}{x^2} + \dfrac{9\frac{1}{4}}{x^3} + \ldots \\ \qquad\qquad \overline{\qquad\qquad\qquad\quad} \end{array}$$

Therefore we find the sum of the roots to be 3½ and the sum of the squares of the roots to be 9¼.

The reason we can divide like this even though the leading coefficient is not unity is because we can always divide numerator and denominator by the leading coefficient (2, in the above example) of the given polynomial to make the leading coefficient equal to 1. Such division will not alter the zeros of the given polynomial.

6. Concluding Remarks

These examples show how straightforward it is to find sums of powers of roots of polynomials by simply applying the *Transpose and Apply* formula of Vedic Mathematics to divide the given polynomial into its 1st differential and pick out the required coefficient(s).

Other sums of combinations of roots can be obtained by dividing the polynomial in question into its 2nd, 3rd etc. differential.

7. Appendix 1

We need to show that for a polynomial of degree n the first differential D_1 = sum of products of $n-1$ factors.

That is, if $f(x_n) = (x + a_1)(x + a_2)(x + a_3) \dots (x + a_n)$ then $f'(x_n)$ = sum of products of $n-1$ factors.

Proof by induction

First, for $n = 2$ we have $f(x_2) = (x + a_1)(x + a_2)$ and for this we see that $f'(x_2)$ = sum of the factors, since $f'(x_2) = f'(x^2 + (a_1+a_2)x + a_1 a_2)) = 2x + (a_1+a_2) = (x + a_1) + (x + a_2)$.

Next assume that for $f(x_k) = (x + a_1)(x + a_2)(x + a_3) \dots (x + a_k)$, $f'(x_k)$ = sum of products of $k - 1$ factors.
We require to show that for:
$f(x_{k+1}) = (x + a_1)(x + a_2)(x + a_3) \dots (x + a_{k+1})$, $f'(x_{k+1})$ = sum of products of k factors.

But $f(x_{k+1}) = f(x_k)(x + a_{k+1})$,

Therefore $f'(x_{k+1}) = f(x_k).1 + f'(x_k)(x + a_{k+1})$

Now $f(x_k)$ is the product of k factors excluding the extra factor $(x + a_{k+1})$.

And $f'(x_k)$ is the sum of the products of $k - 1$ factors, and when multiplied by $(x + a_{k+1})$ becomes the sum of the products of k factors which include $(x + a_{k+1})$.

So together $f(x_k).1 + f'(x_k)(x + a_{k+1})$ is the sum of all products of k factors.

This proves the result.

8. Appendix 2

We wish to show that the coefficients of the power series expansion of $\dfrac{f'(x)}{f(x)}$ in descending powers of x are the number of roots of $f(x)$, the sum of the roots of $f(x)$, the sum of the squares of the roots of $f(x)$, the sum of the cubes of the roots of $f(x)$, etc., in that sequence.

Using the result proved in Appendix 1 we have:

$$\frac{f'(x)}{f(x)} = \frac{\text{the sum of the products of all factors of } f(x) \text{ except the 1st}}{\text{the product of all factors of } f(x)} +$$
$$\frac{\text{the sum of the products of all factors of } f(x) \text{ except the 2nd}}{\text{the product of all factors of } f(x)} +$$
$$\frac{\text{the sum of the products of all factors of } f(x) \text{ except the 3rd}}{\text{the product of all factors of } f(x)} + \dots$$
$$\frac{\text{the sum of the products of all factors of } f(x) \text{ except the last}}{\text{the product of all factors of } f(x)}$$

On cancellation we get:

$$\frac{f'(x)}{f(x)} = \frac{1}{\text{the 1st factor}} + \frac{1}{\text{the 2nd factor}} + \frac{1}{\text{the 3rd factor}} + \dots + \frac{1}{\text{the last factor}}$$

Then if $f(x_n) = (x + a_1)(x + a_2)(x + a_3) \ldots (x + a_n)$:

$$\frac{f'(x)}{f(x)} = \frac{1}{x+a_1} + \frac{1}{x+a_2} + \frac{1}{x+a_3} + \ldots + \frac{1}{x+a_n}$$

$$= \left(\frac{1}{x} + \frac{a_1}{x^2} + \frac{{a_1}^2}{x^3} + \ldots\right) + \left(\frac{1}{x} + \frac{a_2}{x^2} + \frac{{a_2}^2}{x^3} + \ldots\right) + \left(\frac{1}{x} + \frac{a_3}{x^2} + \frac{{a_3}^2}{x^3} + \ldots\right) + \ldots \left(\frac{1}{x} + \frac{a_n}{x^2} + \frac{{a_n}^2}{x^3} + \ldots\right) \quad x \geq 1$$

$$= \frac{\text{no. of roots of } f(x)}{x} + \frac{\text{sum of roots of } f(x)}{x^2} + \frac{\text{sum of squares of roots of } f(x)}{x^3} + \ldots + \frac{\text{sum of } n\text{th powers of roots of } f(x)}{x^{n+1}}$$

as required.

References

[1] Bharati Krishna Tirthaji Maharaja, (1965). Vedic Mathematics. Delhi: Motilal Banarasidas,.

[2] Williams. K. R. (2017). Discover Vedic Mathematics. U.K.: Inspiration Books (page 84).