# Extending the application of Vedic Mathematics sutras James Glover FIMA October 2013

# Abstract

In Chapter 39 of Vedic Mathematics<sup>1</sup> Bharati Krishna mentions in passing, "the rule about Adhyam Antyam and Madhyam bc – ad (i.e. the product of the means minus the product of the extremes)". This is in connection with finding the absolute or independent term for the equation of a straight line. Accepting as hypothesis that the sutras cover all aspects of mathematics, this paper looks at applications of the rule not mentioned by the author.

Research into this curious aphorism, mentioned only once in the book, reveals an interesting and useful approach to several problems involving determinants. The rule has applications relating to one-line solutions for area problems in coordinate geometry, together with one-line proofs of Pythagoras' Theorem and the compound angle formulae. In another context it is also applied to ratio equations and percentage calculations.

The paper shows how a single sub-sutra can be applied to diverse topics in such a way as to unify them.

#### **1. Introduction**

*Vedic Mathematics* by Sri Bharati Krishna Tirtha is an introductory and illustrative text and, as he indicated, there is a vast amount of mathematics which can come under the Vedic sutras<sup>2</sup> which still needs to be discovered. It is therefore befitting to seek applications of the sutras not specifically mentioned in the text. It would be a mistake, however, to suggest that there is necessarily new mathematics coming from the sutras. Much of the mathematics contained in his book can be found elsewhere. But what is unique is the orientation and efficiency of the mathematics. It is based on sutras and these provide a new paradigm for the subject. Students of the subject are encouraged to find the most efficient means of answering problems, aiming towards the *Vedic ideal* of one-line answers and intuitive solutions. A simple example can be seen in the coordinate geometry that students are taught in schools. In conventional text books the equation of a straight line with gradient *m* and one known point,  $(x_1, y_1)$ , is given by  $y - y_1 = m(x - x_1)$ . The Vedic one-line method gives the same equation but in a slightly different form as  $mx - y = mx_1 - y_1$ . The advantages of this are that the equation is obtained in useable form in one line of mental working and it clearly reflects the general (on the left) and particular (on the right) aspects of such equations. In addition it is based on the direct application of the Vedic sutras.

As an example of working with this sutra-based paradigm I have been looking at a little-known sutra that only receives a glancing mention in the book. Chapter 39 of *Vedic Mathematics* is entitled *Analytical Conics* and provides a few specimen examples of how the sutras apply to equations of straight lines. The chapter mentions a rule, or sub-sutra, which, on inspection, can be developed and applied to many problems involving determinants.

#### 2. Equation of a straight line

Bharati Krishna Tirthaji describes the Vedic method for finding the equation of a straight line in the form ax - by = c given the coordinates of two points and using the *Paravartya Yojayet* sutra. The coefficients *a* and *b* are obtained from the difference between the *y* and *x* coordinates respectively, subtracted in the same direction. He then describes three methods for obtaining the independent term, *c*. The first two of these are by substitution of the known coordinates into the left-hand side of the equation. The third method is where he mentions,

"the rule about Adyam, Antyam and Madhyam, i.e. bc - ad, i.e. the product of the means *minus* the product of the extremes."

When the coordinates are set out in a row, for

example (3,8) (2, 4), (Figure 1) the equation of the line passing through them is,

 $(8-4)x - (3-2)y = 8 \times 2 - 3 \times 4$ 

which simplifies to 4x - y = 4



Figure 1

The right-hand side of the equation is obtained by applying the rule. The word *means* refers to the middle two numbers, 8 and 2, whilst extremes applies to the outer numbers, 3 and 4. The fact that in the text the word *minus* is in italics suggests that this is not contained in the sutra itself but is implied by context, and that there are other applications in which the word equals or plus may be applicable. For example, if the rule reads, The product of the means equals the product of extremes, then it is immediately applicable to all problems involving simple proportion. For wherever there is an equality of ratio, such as 3:4=9:12, it can be seen that the product of the inner two numbers is always equal to the product of the outer two numbers, that is  $4 \times 9 = 3 \times 12$ . This provides a reliable approach to rendering any such ratio problem, usually in

the form of words, into an equation that can then be solved quite easily.

The Adyam rule is only mentioned once in the whole book and so we have to judge what the wording of the sub-sutra is from the various clues. The word *Adyam* means first and *Antyam* means final and here the connection is one of multiplying. The word *Madhyam* means middle. The sense of the rule is, *The first by the last* (in other words, the product of the extremes), *the middle by the middle*. This is very similar to another sub-sutra, *Adyamadyena Antyamantyena*, which he translates as, *The first by the first and the last by the last*<sup>3</sup>, and is used in connection with factorizing polynomials.

Putting this together and following in the same vein the proposed wording of this sub-sutra is, *Adyamantyena Madhyamadhyena*, literally, *the first by the first – the middle by the middle*, and which can be rendered as,

The product of the extremes and the product of the means.

When the word *minus* is introduced then the sutra reveals a determinant. For example, with the coordinates (3,8)

(2, 4), as mentioned before, when these are placed into matrix form as

 $\begin{pmatrix} 3 & 2 \\ 8 & 4 \end{pmatrix}$  or  $\begin{pmatrix} 2 & 3 \\ 4 & 8 \end{pmatrix}$ , the positive value of the

determinant is the same as

 $2 \times 8 - 3 \times 4 = 16 - 12 = 4$ .

#### 3. Parallelograms

A simple application of the rule can be found in the following problem:

A parallelogram has one vertex at the origin and the two adjacent corners have coordinates (3,8) and (5, 2) (Fig 2). The problem is to find the area of the parallelogram. Application of the sutra immediately gives  $8 \times 5 - 3 \times 2 = 34$  and since this is achieved by simple mental arithmetic meets the Vedic ideal<sup>4</sup> of one-line working. If the coordinates are placed the other way round, (5, 2), (3,8) then it just needs to be remembered that the positive value is required.



Figure 2

What happens when no corner is on the origin such as in the following problem? Find the area of the parallelogram

(Figure 3) with vertices at (2, 3), (8, 5) and (1, 9). Note that only three vertices need to be known. This is solved using the *Paravartya* sutra and transposing the parallelogram so that one vertex is at the origin. This is done by subtraction.





The translation produces a new set of coordinates for the two vertices as (6, 2) and (-1, 6). The area formula can then be applied,

$$2 \times (-1) - 6 \times 6 = 38$$

The conventional method requires working out the determinant of a 3 by 3 matrix,

$$\begin{vmatrix} 2 & 3 & 1 \\ 8 & 5 & 1 \\ 1 & 9 & 1 \end{vmatrix}$$
  
= 2(5×1-1×9)  
-3(8×1-1×1)  
+1(8×9-5×1)  
= -8-21+67 = 38

The advantage with this "Vedic" method over the conventional method is that it is very simple to remember and also reduces the number of arithmetic operations involved by a half.

## 4. Triangles

Since every triangle can be considered as half a parallelogram the sutra can be applied to problems of finding areas of triangles.



In the example above, the area is found by halving the area of the parallelogram.

 $\frac{1}{2}(8 \times 7 - 4 \times 3) = 22$ .

A converse problem is to find a third point of a triangle given two points and its area.



Figure 5

In Figure 5, a triangle has one vertex at the origin and another at (8, 2). Given that the area is 16 square units, find a possible position for the third vertex,

(x, y).

From the area formula,

$$\frac{1}{2}(8y-2x) = 16,$$
  
simplifying to  $4y - x = 16$   
or  $y = \frac{1}{4}x + 4$ 

So the third vertex can lie anywhere on this line

which is parallel to the opposite side of the triangle. A value for x can be chosen and the corresponding value for y is calculated from the equation, for example (4, 5).

#### 5. How does the area formula work?

The method can be explained from the point of view of matrices. When a 2 by 2 transformation matrix is applied to a shape on a graph the determinant gives the *area scale factor*<sup>5</sup>. When applied to the unit square the resulting shape, which, for non-zero determinants will always be a parallelogram, has an area equal to that determinant.



In Figure 6, the unit square is mapped to the parallelogram with (1, 0) mapped to (6, 1) and (0, 1) mapped to (2, 3).

$$\begin{pmatrix} 6 & 2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 6 & 2 \\ 1 & 3 \end{pmatrix}$$

The determinant of the transformation matrix is  $6 \times 3 - 2 \times 1 = 16$  and this is the area scale factor. The parallelogram is therefore 16 times the area of the unit square, in other words, 16 square units.

# 6. Co-linearity

This method can also be used to find out if three points are collinear because, if they are, then the area of the parallelogram formed will be zero. Here is an example. Find whether the points (1, 3), (5, 5) and (13, 9) are collinear. Assuming that this forms three vertices of a parallelogram we first translate so that one vertex, say (1, 3), is moved to the origin. As before this is done by subtraction.

$$(5,5) (13,9) 
-(1,3) (1,3) 
(4,2) (12,6)$$

The remaining points are then at (4, 2) and (12, 6). The area formula then gives  $2 \times 12 - 4 \times 6 = 0$ . This means that the area of the parallelogram is zero and hence the three points are collinear.

## 7. Pythagoras' theorem

This application, for finding areas of parallelograms, gives the basis of a one-line proof of Pythagoras' theorem. Consider a square of side *c* and area  $c^2$ , with one vertex at the origin (Figure 7). The square is rotated anticlockwise so that one vertex lies at *(a, b)* and it necessarily follows that the opposite vertex lies at *(-b, a)*. Using the sutra, the area of the square is  $a^2 + b^2 = c^2$ .



## 8. Compound Angle Formulae

A further application is in establishing compound angle formulae for sine and cosine.

Consider the parallelogram OPQR

(Figure 8) with unit sides OR and OP making angles of A and B with the positive x-axis, respectively. The coordinates of R and P are then  $(\cos A, \sin A)$  and  $(\cos B, -\sin B)$ .





parallelogram,

Area = 
$$\sin A \cos B + \cos A \sin B$$

But by using the formula for the area of a triangle (involving half the product of two sides and the sine of the angle between them), which can be doubled to find the area of the parallelogram,

Area OPQR = 
$$2 \times \frac{1}{2} \times 1 \times 1 \times \sin(A+B)$$

Hence,  $\sin(A+B) = \sin A \cos B + \cos A \sin B$ 

The diagram below (Figure 9) is used for the compound angle (A - B).

Since OR and OP make angles *A* and *B*, respectively, with the positive x-axis then angle ROP = (A - B).

Again, using the half sine formula for the area of a triangle,

Area =  $2 \times \frac{1}{2} \times 1 \times 1 \times \sin(A - B)$ 





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Area =  $2 \times \frac{1}{2} \times 1 \times 1 \times \sin(A - B)$ 

Using the sutra,

Area OPQR =  $\sin A \cos B - \cos A \sin B$ 

Hence,  $\sin(A-B) = \sin A \cos B - \cos A \sin B$ .

Similar simple proofs can be constructed for dealing with the cosine of compound angles.

# 9. Vectors

The Madhyamadhyena Antyamantyena rule, *The* product of the means minus the product of the extremes, together with the Adhyamadhyena Antyamantyena rule, *The first by the first and the* last by the last, form the basis of vector multiplication because they are identical to the vector product and scalar product, respectively. The vector product of two vectors **u** and **v** is defined as  $\mathbf{u} \times \mathbf{v} = uv\sin\theta.\mathbf{n}$ , where uv is the product of the lengths of vectors **u** and **v**,  $\theta$  is the angle between them and **n** is the unit vector perpendicular to both.

Taking account of the right-hand rule for vector products we let  $\mathbf{u}$  be a position vector with end point at (*c*, *d*) and  $\mathbf{v}$  with end points (*a*, *b*).





The parallelogram formed with **u** and **v** has area base times height, which is  $uv\sin\theta$ . Using the area formula this is also bc - ad. Hence  $\mathbf{u} \times \mathbf{v} = (bc - ad).\mathbf{n}$ .

The dot product is obtained by, *The first by the first and the last by the last.* 

The two vectors, u and v can be written as,

$$\mathbf{v} = \begin{pmatrix} a \\ b \end{pmatrix} \text{ and } \mathbf{u} = \begin{pmatrix} c \\ d \end{pmatrix}$$

The dot product is then ac + bd.

So these two sutras provide the modulus of the vector product and the vector scalar product and, as such, forms the basis of many aspects of calculation and mathematics used in engineering<sup>6</sup>.

## **10.** Conclusion

It is hoped that this paper provides an example of how the sutras can be extended and utilized to branches of mathematics not specifically mentioned in Sri Tirthaji's book. The process requires taking each sutra as the principle to work with as a base and then seeing how it can be applied in the various branches of mathematical activity. As mentioned previously this produces a change in paradigm so that mathematics is seen from the point of view of the Vedic sutras. Undoubtedly the result is a more unified approach because a single sutra has multifarious applications and, through them, different topics and branches of mathematics become connected.

#### **Bibliography**

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<sup>3</sup> *ibid*, page 74
<sup>4</sup> *ibid*, page 72
<sup>5</sup> http://math.stackexchange.com /questions/29128/why-determinant-of-a-2-by-2matrix-is-the-area-of-a-parallelogram
<sup>6</sup> http://www.netcomuk.co.uk/~jenolive /vect8.html