A NEW APPROACH TO SOLVING LINEAR EQUATIONS

Ashwini Kale, Dr. Anant Vyawahare

1. Introduction

Shri Krishna Daivagna (16-17thC), a profound scholar in Mathematics, wrote an excellent commentary on Bijaganit of Bhaskara II and called it *Bijapallavam* (Sprout of Algebra) [1].

Aryabhata II (920-990 AD), in his eminent work *Mahasiddhanta* [2], gave general as well as specific rules for solutions of equation $ax \pm c = by$ where a, b, c are positive constants.

The present work suggests a new method of solving linear equation $ax + c = by$, stressing the importance of constant c.

Initially, solution of this equation is obtained for positive values of a, b, c by considering different conditions over an additive c.

2. Use of Augmentation *c*

Consider a linear equation $ax + c = by$, where $a \neq 0$, $b \neq 0$ and a, b, c are all positive integers, c is called *augmentation*. Here we consider the effect of c on the solution of above equation.

Case I When $c = 0$.

Here the solutions are

(i) $x = y = 0$, a trivial solution,

(ii) $x = b$, $y = a$ as $ab + 0 = ba$.

The second solution can be generalised as $x = kb$ and $y = ka$ with k is an integer.

Case II When $a - b = 1$, $c \neq 0$.

Here $a = b + 1$. Hence $(b+1)\, x + c = by$.

By observation, $x = y = -c$ is its solution.

Now we have two solution sets:

$x = tb$ & $x = -c$, and $y = ta$ & $y = -c$

Hence general solution is $x = tb - c$ & $y = ta - c$

For $a = b + 1$, we get $x = tb - c$ & $y = t\,(b+1) - c$, $t = 0,1,2.......$

This can be illustrated by example $4x + 10 = 3y$.

Here, the solution at $t = 1$ is $x = b - c$ & $y = b + 1 - c$ i.e. $(x, y) = (-7, -6)$.

Other solutions, for varying t, are (-10, -10), (-4, -2), (-1, 2), (2,6)......

Case III When $a = bq + r$, $(q = 1)$, $r \neq 1$, r divides c i.e. $c = rq_1$,

If $x = r$, $y = r$ is a solution of $ax - c = by$ then

$x = b - q_1$, $y = a - q_1$ is a solution of $ax + c = by$.

For $17x + 6 = 15y$,

$17 = 15(1) + 2$ & 2 divides 6 such that $q_1 = 3$,

hence $(x, y) = (12, 14)$.

Case IV When $a = bq + r$, $(q \neq 1)$, $r \neq 1$, r divides c i.e. $c = rq_1$,

(i) Assume $x = b - kq_1$, $y = a - kq_1$, k is an integer.

Then, $ax + c = by$ gives $k = r/(a-b)$.

When $q = 1$, $r = a - b$ then $k = 1$.

For $16x + 8 = 7y$,

$16 = 7(2) + 2$ and 2 divides 8 such that $q_1 = 4$.

here, $x = 7 - 4k$ & $y = 16 - 4k$ where $k = 2/9$.

Therefore $(x, y) = (55/9, 136/9)$.

(ii) Let $x = X$ & $y = X + c$ for some real number X, then, $ax + c = by$ gives

$X = (b - 1)c / (a - b)$.

For $16x + 8 = 7y$, $x = 16/3$ & $y = 40/3$ satisfying the above equation.

Case V When $a = bq + r$, $(r \neq 1)$, r does not divide c $(q = 1)$.

Here, quotients $q_1, q_2, q_3, q_4 \ldots$ etc. obtained after continuous division, are written one below the other in the form of column followed by an additive c. The last entry of the column is zero. Later the second to last number of the column is multiplied by the number above and added it to the number below while the last number is discarded.

For $19x + 7 = 17y$,

$$\begin{array}{l} 17\overline{)19}(1 \\ \quad 17 \\ \quad \overline{02})17(8 \\ \qquad 16 \\ \qquad \overline{01} \end{array}$$

Quotient & constant column is

1	63 = Y
8	56 = x
7	

Here $x = 8 \times 7 = 56$, $y = 8 \times 7 + \mathbf{7} = 63$. Hence

(i) The first integer solution is (56 , 63).

(ii) A least positive solution is obtained by

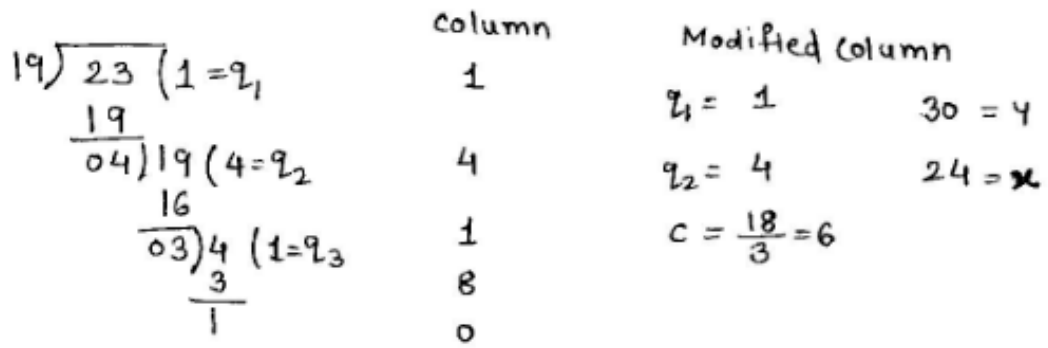
$$\left.\begin{aligned} x &= 56 - 17k \\ y &= 63 - 19k \end{aligned}\right\} \text{ for maximum } k \text{, such that } x \text{ and } y \text{ are positive}$$

For $k = 3$, $x = 5$ & $y = 6$. Thus (5,6) is the least positive solution of the equation.

(iii) For $k = 1,2,\ldots..$, (x, y) : (5,6), (22,25), (39, 44), (56, 63) .... are the infinite solutions as the graph of equation is a straight line.

Case VI When the first remainder after division of a by b does not divide c but the remainder obtained at later steps divides c completely.

For $23\,x + 18 = 19\,y$,



Here, the first remainder 4 does not divide 18 but the second remainder 3 divides 18 completely. In such a case the division may be stopped where the remainder divides c completely and 18/3 = 6 is considered as a new additive c. Then the solution of this equation is (24, 30) (but removing a common factor of 6, the new pair (4,5) is not a solution ! The reason is that if (x, y) is a solution, $(x/k, y/k)$ may not lie on the graph.)

Case VII To solve $ax + c = by$ using quotients, where $a > b$ and x, y are real numbers.

Suppose the continuous division gives quotients, $q_1, q_2, q_3, q_4 \ldots..$ etc. until the last remainder is zero.

Let Q = product of all these quotient $= q_1 \cdot q_2 \cdot q_3 \cdot q_4 \ldots..$

Assume that $x = Q$ and $y = Q + k$ for some real number k.

then $ax + c = by$, gives $k = [Q(a - b) + c] \;/\; b$. This will lead to a final solution.

For $16x + 8 = 7y$,

$16 = 7(2) + 2$, $7 = 2(3) + 1$,.. hence, $Q = 2.3.2 = 12$ such that $2 = 1(2) + 0$
then $k = 116 / 7$. Hence the solution is $(x, y)= (12, 200/7)$

3. Use of Negative divisors

When the divisor b is negative, y will be negative.

Case I Generally, negative divisors are neglected, as the remainder becomes larger than the divisor which contradicts the division algorithm.

For $16 \div (-3)$, $16 = (-3)(5) + r$. Here $r = 31 > 16$.

When b is negative, y will be negative .When a and b have opposite sign, the division gives a negative quotient.

For $(-15) / 5 = -3$, hence y will be negative.

For $15x + 7 = -13y$

If all a, b, c are kept positive i.e. $15x + 7 = 13y$, then (42,49) is the solution and least positive solution is (3, 4).

The solution of $15x + 7 = -13y$ will be (3, -4).

Thus, if (x_1, y_1) is a solution of equation $ax + c = by$, then $(x_1, -y_1)$ is solution of

$ax + c = -by$, and conversely.

Case II- Solution of $ax \pm c =$ by using Continued Fractions.

Initially the solution of equation $ax \pm 1 = by$, is obtained by continued fraction and then extended it for $ax \pm c = by$.

For $15x + 8 = 13y$, initially consider $15x + 1 = 13y$.

Then $\frac{15}{13} = 1 + \frac{2}{13} = 1 + \frac{1}{\frac{13}{2}} = 1 + \frac{1}{6+\frac{1}{2}} = 1 + \frac{1}{6+\frac{1}{\frac{2}{1}}}$ etc.

Here $15(6) - 13(7) = -1$, Hence (6, 7) is the solution of equation $15x + 1 = 13y$.

Then for equation $15x + 8 = 13y$, the solution is $x = 13t + 48$ & $y = 15t + 56$ for

$t = 0,1,2...$

4. Odd and Even Number of Quotients

The sign of a, b, c in $ax + c = by$ will decide if the number of quotients in division algorithm are even or odd . These are useful in finding the g c d 'd' of a & b such that

$d = ap + bq$ for some integers p & q.

4a For odd number of quotients:

Case I *a*, *b*, *c* are all positive.

For 86 x + 4 = 13y,

	Column	
$13)\,86\,(6 = q_1$	6	132
78		
$08)\,13\,(1 = q_2$	1	20
8		
$5)\,8\,(1 = q_3$	1	12
5		
$3)\,5\,(1 = q_4$	1	8
3		
$2)\,3\,(1 = q_5$	1	4
2	4	
1	0	

Here (20, 132) is not solution of above equation, i.e. first two entries of the column do not give solution when number of quotients are odd .

1st reduction gives y = 132 - a = 46 & x = 20 - b = 7, which is not the solution.

2nd reduction gives y = a - 46 = 40 & x = b - 7 = 6 , which is a solution of given equation.

Thus, when a, b, c are positive and number of quotients are odd, then 2nd reduction is necessary to obtain the solution of equation.

Case II *a* is negative:

For -86 x + 4 = 13 y, (20, -132) is not the solution.

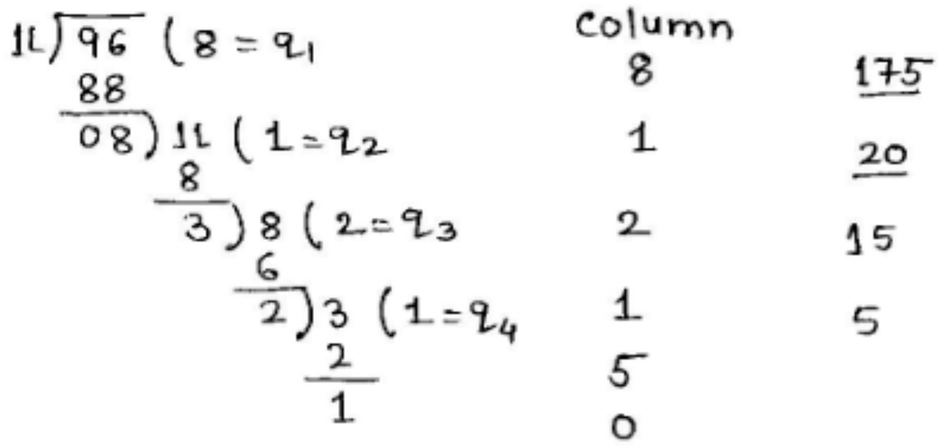
1st deduction gives (7, -46) which is solution of given equation.

Thus, when a is negative and number of quotients are odd, second reduction is not necessary only first reduction will give the solution of equation.

4b For an Even Number of Quotients:

Case I ***a*, *b*, *c* are all positive.**

For $96x + 5 = 11y$,



Here (20, 175) is solution of given equation.

Thus when *a*, *b*, *c* are all positive and number of quotients obtained when *a* divided by *b* are even then solution can be obtained directly.

Case II ***a* is negative**

For $-96x + 5 = 11y$,

Here (-20, 175) is solution of the equation.

Thus, when *a* is negative and number of quotients are even then without any reduction we get solution, here only sign of *x* is negative whereas *y* remains same.

Case III ***b* is negative:**

For $96x + 5 = -11y$, (-20, 175) is not a solution.

Here (-2, 17) is a solution of the given equation.

Hence, when *b* is negative and number of quotients are even, then second reduction is necessary to obtain the solution.

Thus equation can be solved for different values of *a*, *b*, *c* (all positive or *a* negative or *b* negative), necessary reduction should be done to obtain solution of equation depending on number of quotients.

5. Remarks

The methods discussed in this paper to solve equation $ax + c = by$ are easy, involve less number of steps and gave least integer solution in few steps. The advantages of these methods are

(i) Stress on the importance of constant c

(ii) Unique approach to solutions using constant c.

(iii) Use of negative divisors.

(iv) Solution using continued fractions.

(v) Solutions for nature of quotients.

References

[1] Dr. Sita Sundar Ram, *Bijapallava of Krsna Daivajna*,

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[2] Vedveer Arya , *Indian Contributions to Mathematics & Astronomy*,

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