**MULTIPLICATION BY NINES AND ITS ASTRONOMICAL APPLICATIONS**

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**Abstract**

In this paper an astronomical application of Tirthaji's method for multiplying by numbers consisting solely of nines is shown. This method is then extended to deal with multiplications where the multiplier consists of a series of nines preceded by another digit, and a further astronomical application is shown using this. How this type of multiplication can be used in positional astronomy, in which a body's orbit is described with reference to the celestial equator, is then shown.

## 1. Introduction

In Chapter 2 of his book[1] Bharati Krishna Tirthaji discusses multiplications in which the multipliers consist of a series of nines. He writes "we shall just now explain a few corollaries which arise out of the '*Nikhilaṁ*' Sūtra which is the subject-matter of this chapter." Then follow three corollaries, the third of which is our concern here.

Under this third corollary Tirthaji writes "Then comes a Third Corollary to the *Nikhilaṁ Sūtra*, which relates to a very special type of multiplication and which is not frequently in requisition elsewhere but is often required in mathematical astronomy etc. The wording of the sub-sūtra (corollary) *Ekanūnena Pūrveṇa*, sounds as if it were the converse of the *Ekādhika* Sūtra. It actually is; and it relates to and provides for multiplications wherein the multiplier-digits consist entirely of nines."

This section is divided into three 'cases' depending on whether the number of nines in the multiplier is the same as, more than, or less than, respectively, the number of digits in the multiplicand.

At the end of the first case Tirthaji makes another reference to Astronomy: "Such multiplications (involving multipliers of this special type) come up in advanced astronomy etc. and this sub-formula (*Ekanūnena Pūrveṇa*) is of immense utility therein."

The first two cases will be described in the next section of this paper, followed by a section that shows how this may be used in Astronomy. Then we show a simple development of this type of multiplication to deal with multipliers consisting of a series of nines preceded by another digit, e.g. 199, 3999 etc. and show an astronomical application of that.

## 2. Multiplication of Numbers of the Form $10^n \pm 1$

Multiplication by numbers like 101, 1001, 10001 etc. is very easy as the multiplicand simply gets repeated. For example:

$74 \times 101 = 7474$,
$748 \times 1001 = 748748$,
$74 \times 1001 = 74074$, and so on.

For products like 77 × 99, 798 × 99999 etc. Tirthaji shows a simple application of the Nikhilaṁ Sūtra (*All from 9 and the Last from 10*), as shown below.

Example 1: 77 × 99

$$\begin{array}{l} 77 - 23 \\ \underline{99 - \ \ 1} \\ \underline{76 \,/\, 23} \end{array}$$

We see from this application of base multiplication that the product is found by:
1) reducing the multiplicand by one i.e. 77 – 1 = 76,
2) and applying *All from 9 and the Last from 10* to the multiplicand i.e. 77 becomes 23.
This gives us 77 × 99 = 7623.

Example 2: 9879 × 9999 = 9878/0121.

Example 3: 798 × 99999.

In this example we have 3 digits in the multiplicand and 5 digits in the multiplier. We deal with this by simply placing (or imagining) two zeros prefixed to 798:

798 × 99999 = 00798 × 99999 = 00797/99202 = 79799202.

An extension of this method is shown later, where it is also proved.

**3. Pythagorean Triples**

We may define a Pythagorean triple as a set of three numbers *a*, *b*, *c* for which $a^2 + b^2 = c^2$ and *a*, *b*, *c* are integers.

Examples of these are:

3, 4, 5
-21, 20, 29
99, 20, 101

There are many fascinating properties amongst these triples and they have been the subject of extensive research by many people over many centuries.

Of particular interest for us is a family of triples exemplified by their first member:

99, 20, 101.

We may notice here that the square of half the middle element is 100, and that the first and last elements are 1 below 100 and 1 above 100 respectively.

This family may be extended as follows:

| | |
|---|---|
| 99, 20, 101 | angle 0.2 radians |
| 9999, 200, 10001 | angle 0.02 radians |
| 999999, 2000, 1000001 | angle 0.002 radians |
| etc. | |

In terms of triangles, we define the three elements of a triple to be the base, height and hypotenuse respectively of the triangle they describe, and we define the angle in a triple as that between the base and the hypotenuse.

We see some clear patterns in this family of triples; and their angles, given approximately above in radians, are memorable. That approximation becomes closer as we go down the list of triples.

## 4. Astronomical Application

In Astronomy we often deal with small angles and so it is not surprising that we can make use of these triples here (and other classes of triples too).

Also in Astronomy, positions are given by angles in certain planes and these angles are normally defined by trigonometrical functions before being converted to an angle in degrees or radians.

The longitude, *L*, of a planet for example may be defined by cos L = 0.8 and this is equivalent to the triple:

$$L)\ 4, 3, 5$$

Because a triple defines an angle, it is possible to work entirely with triples and thereby avoid the general incommensurability between angles and their trig functions.

### 4.1 Triple Addition

It is therefore convenient to be able to add and subtract triples in such a way that the angles in the triples get added and subtracted.

Such definitions of triple addition and subtraction are given in my book "Triples"[2].

Triple addition for example is defined by:

| | | | | |
|---|---|---|---|---|
| $X$ | $a$ | $b$ | $c$ | |
| $Y$ | $d$ | $e$ | $f$ | + |
| $X+Y$ | $ad-be$ | $bd+ae$ | $cf$ | |

where $a$, $b$, $c$ and $d$, $e$, $f$ are triples containing angles $X$ and $Y$ respectively.

The result is a triple with an angle of $X+Y$, and it is obtained using the *Vertically and Crosswise* Sutra.

Now we can show an application of the method of multiplying by numbers of the form $10^n\pm1$ in Astronomy.

**4.2 Example 4**

Suppose a planet has longitude given by the triple *L*)84,13,85. Find a triple for its new longitude when it has moved through an angle of 0.2 radians.

We simply add the triples for *L* and 0.2 as shown below:

| | | | | |
|---|---|---|---|---|
| *L*) | 84 | 13 | 85 | |
| 0.2) | 99 | 20 | 101 + | |
| *L*+0.2) | 84×99–13×20 | 13×99+84×20 | 85×101 | using the *Vertically and Crosswise* |
| = | 8316–260 | 1287+1680 | 8585 | using the special multiplications |
| = | 8056 | 2967 | 8585 | |

**4.3 Example 5**

Similarly given *L*) 15,8,17 we can find the new position after an increase in position of 0.02 radians as follows:

| | | | |
|---|---|---|---|
| *L*) | 15 | 8 | 17 |
| 0.02) | 9999 | 200 | 10001 + |
| *L*+0.02) | 15×9999–8×200 | 8×9999+15×200 | 17×10001 |
| = | 149985–1600 | 79992+3000 | 170017 |
| = | 148385 | 82992 | 170017 |

As mentioned in the Introduction, Tirthaji does refer to this method of multiplication by nines as "a very special type of multiplication", which means it applies in only certain cases. The application shown here is for dealing with angles of 0.2, 0.02, 0.002 etc. radians. But we may extend this range of application as shown in the next section.

**5. Extension of the Multiplication by Nines Method**

Let us consider 777 × 1999. Here we have a series of nines in the multiplier but they are preceded by a '1'.

In fact 777 × 1999 = 1553 / 223
where we multiply the multiplicand by the number one more than the one before the series of nines. That is we multiply 777 by 2 to get 1554, and then reduce this by 1, as before, to get 1553 (the left-hand part of the answer). The right-hand side is, as before, obtained by applying *All from 9 and the Last from 10* to the multiplicand, 777, to get 223.

Proof: Suppose $m10^n - 1$ represents a number 1 below a multiple of a power of 10. Let the multiplicand be *N*, then:
$N(m10^n - 1) = (Nm - 1)10^n + (10^n - N)$, the right-hand side expressing the method used above.

Similarly 76 × 299 = 76×3–1 / 24 = 227/24.

### 5.1 Triples

This type of multiplication can be useful when dealing with the following class of triples:

399, 40, 401 angle 0.1 radians
39999, 400, 40001 angle 0.01 radians
3999999, 4000, 4000001 angle 0.001 radians
etc.

### 5.2. Astronomical Application - Example 6

A planet has longitude given by *L*)45,28,53. Find a triple for its longitude when it has moved through an angle of 0.1 radians.

| | | | | | |
|---|---|---|---|---|---|
| *L*) | 45 | 28 | 53 | | |
| 0.1) | 399 | 40 | 401 | + | |
| *L*+0.1) | 45×399–28×40 | 28×399+45×40 | 53×401 | | using the *Vertically and Crosswise* |
| = | 17955–1120 | 11172+1800 | 21253 | | using the special multiplication |
| = | 16835 | 12972 | 21253 | | |

## 6. Quadruples: 3-Dimensions

### 6.1 Introduction

Astronomers define positions in space using an equatorial rectangular coordinate system, and another way in which this multiplication by nines comes about in Astronomy is in finding planetary positions in such a system.

The beautiful way in which triples expand into 3-dimensional space (we call the 3-dimensional equivalent of triples 'quadruples') and beyond, and in which quadruples collapse into triples will have to be the subject of another paper, but a brief description of this, and the way we can add these quadruples while making use of multiplication by nines, will be given here.

### 6.2 The Generating Formulae for Triples and Quadruples

Triples can be generated by $c^2 - d^2$, $2cd$, $c^2 + d^2$ where $c, d \in$ ¢ .

In the 3-dimensional equivalent, 'quadruples' are generated by:
$c^2 - d^2 + e^2$, $2cd$, $2de$, $c^2 + d^2 + e^2$ where $c, d, e \in$ ¢ . [1]

Note the similarities in these two generating formulae.

If for example $c$=3, $d$=1 and $e$=2 we generate the quadruple 12,6,4,14, which simplifies to 6,3,2,7, by dividing out the common factor, 2.

If these quadruples are described by $x,y,z,r$ then we have the property that $x^2+y^2+z^2 = r^2$ and this can be proved by showing the the sum of the squares of the first three elements in [1] is equal to the square of the fourth. Thus in the numerical example $6^2+3^2+2^2 = 7^2$.

### 6.3 Astronomical Application of Quadruples

Geometrically (6,3,2) could be the coordinates of a point in 3-dimensional space and 7 will be the distance of that point from the origin. It can be represented by a pyramid consisting of four right-angled triangles as shown in Figure 1.

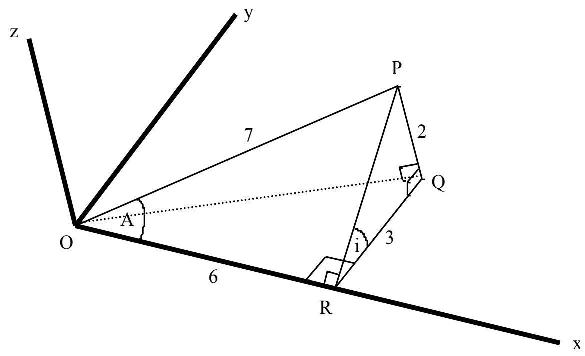


Figure 1

Here the $x$-$y$ plane, in which triangle ORQ lies, will be the reference plane, which will be the celestial equator (that is the projection of the Earth's equator onto the celestial sphere).

The plane defined by triangle ORP will be a plane inclined to this equatorial plane at an angle $i$ as shown. P moves in this orbit plane with $i$ constant but with angle A increasing.

It is important to note that in 6,3,2,7 the middle two elements, 3,2, define the orbit inclination, while the two outer elements, 6,7, define A, the angle in the quadruple.

We may wish to find the equatorial coordinates of P once it has moved through a given angle in its orbit plane.

### 6.4 Example 7

The equatorial rectangular coordinates of a planet at a certain instant are given by 12,4,3,13. Find its equatorial coordinates when it has moved in its orbit through 0.2 radians.

We know a triple containing the given angle: 0.2)99,20,101.

We convert this to a quadruple and add the result to the given quadruple.

The conversion requires us to leave the 99 and 101 as they are but expand the 20 into two values, say $p$ and $q$, such that $p:q = 4:3$ (as in the given quadruple 12,4,3,13) and $p^2 + q^2 = 20^2$.

This gives us 99,16,12,101 and is actually very easy to do in practise because:
the 3rd element of the triple 4,3,- is 5,

and to get 20 instead we must multiply 5 by 4.
We therefore multiply 4,3 by 4 to get 16,12.

Now we add the quadruples:

| | | | | | |
|---|---|---|---|---|---|
| *L*) | 12 | 4 | 3 | 13 | |
| 0.2) | 99 | 16 | 12 | 101 + | |
| *L*+0.2) | 12×99–4×16–3×12 | 4×99+12×16 | 3×99+12×12 | 13×101 | [2] |
| = | 1088 | 588 | 441 | 1313 | |

Quadruple addition, defined by line [2] above, is very similar to triple addition and has the effect of adding the angles but keeping the same orbit plane[2].

The *Vertically and Crosswise* pattern is again in operation here.

Notice that 588:441 = 4:3.

The new equatorial coordinates are (1088,588,441,1313).

We see the application of multiplying by nines here and of course many other angles (like the 0.2 radians here) can be catered for.

## 7. Concluding Remarks

There are other families of triples that involve series of nines and so applications need not be restricted to angles of the form $0.2 \times 10^{-n}$ radians or $0.1 \times 10^{-n}$ radians.

Converting between radians and degrees, if required, may be facilitated by convenient approximate equivalents, of appropriate levels of accuracy, such as 4 degrees ≈ 0.07 radians.

## References

[1] Bharati Krishna Tirthaji Maharaja, (1965). Vedic Mathematics. Delhi: Motilal Banarasidas,.

[2] Williams K. R. (2013). Triples. U.K.: Inspiration Books