BY MERE OBSERVATION

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Abstract

Of Vedic mathematics sutras Vilokanam appears to be the odd one out. All the other sutras have a connection with a logical process but this one connects us with areas of the human psyche related to perception and intuition. It points to a more subjective and personal approach to problem solving. This paper reviews references to Vilokanam in Bharari Krishna Tirtha's book and attempts to describe the nature of perception and intuition in relation to mathematics. It concludes with a discussion on the subjective and objective nature of Vedic mathematics.

1. How Vilokanam is used in Vedic Mathemaitcs

In the Vedic Mathematics book Vilokanam appears in the list of sub-sutras and there are twelve references in the text. BKT refers to this sutra in cases where the solution to a problem is found just by looking.

There is one reference to what is perhaps the full wording, Vilokanenaiva, but all others state "Vilokanam". The word *Vilokanam* is made up of the prefix *Vi* and the noun *lokanam*. The root is *Lok* from which we derive the English word "look". The grammatical ending on Vilokanenaiva has the meaning, "by means of observation only".

The first three are in the chapter on Paravartya Division.

a) Divide 1234 by 160

Here BKT states that Nikhilam division "is manifestly unsuitable" and then proceeds with the Paravartya method. It is set out like this,

$$\begin{array}{r|rr}
1\,6\,0 & 1\;2 & 3\;4 \\
\bar{6}\,0 & \bar{6} & 0 \\
 & & 240 \\
\hline
 & 6 & 274 \\
\cline{2-3}
 & 1 & -160 \\
\cline{2-3}
 & 7 & 114
\end{array}$$

To explain the method, digits other than the 1 of 160 are transposed and used as multipliers by the quotient digits. The first step of the answer is 6 remainder 270. Since the remainder is larger than the divisor he adds 1 to the quotient and subtracts 160 from the remainder giving 7 remainder 114 as the final answer. Unfortunately, it is not clear where the 240 has come from and this may be due to some error in typography. In applying Paravartya as described in the chapter it would normally be set out as follows:

$$
\begin{array}{r|rrr}
160 & 1 & 2 & 3\ 4 \\
\bar{6}\,0 & & \bar{6} & 0 \\
 & & & 240 \\
\hline
 & 1 & \bar{4} & 274 \\
\hline
 & & 6 & 274 \\
\hline
 & & 1 & -160 \\
\hline
 & & 7 & 114
\end{array}
$$

With this setting it is clear that the 240 has come from multiplying $\bar{4}$ by $\bar{6}\,0$.

He then adds the comment, “But this is a case where (Vilokanenaiva) i.e. by simple inspection or observation, we can put the answer down”.

If the reader is sufficiently familiar with the 16 times table and intuitively knows that 160 times 7 is 1120 then, yes, on subtracting 120 from 234 to give 114 for the remainder, the answer can be achieved by observation. However, unless this is known then the answer cannot be arrived at so easily. This points to the subjective nature of the sub-sutra which will be discussed later.

The next two referals ocurr in the same chapter.

b) For 11329 divided by 1132, he shows the Paravartya method but one can immediately see that the answer is 10 remainder 9.

c) The following example is 103 divided by 82. Again it can easily be seen that, on adding 18 (the deficiency of 82 from 100) to 3, the answer is 1 remainder 21.

Interestingly, he uses the Paravartya method, even though Nikhilam would be more suitable, as follows:

$$
\begin{array}{r|rrr}
82 & 1 & 0 & 3 \\
1\bar{2}\,2 & & 2 & \bar{2} \\
2\,\bar{2} & & & \\
\hline
 & 1 & /2 & 1 \\
\hline
\end{array}
$$

So he has first written 82 in vinculum form as $1\bar{2}\,2$ and then transposed the two final digits. This demonstrates the flexibility of the Vedic approach.

d) In chapter 14 on Complex Mergers we see the next reference.

It is in relation to one example,

$$\frac{1}{2x-1}+\frac{8}{4x-1}=\frac{6}{3x-1}+\frac{3}{6x+1}$$

This is immediately rearranged as,

$$\frac{6}{12x-6}-\frac{6}{12x-2}=\frac{24}{12x-4}-\frac{24}{12x-3}$$

I will not go into the machinations of the formula for complex mergers but point out that he then concludes that $x=0$ and comments, "Vilokana (i.e. mere observation) too will suffice in this case". Again this indicates that Vilokanam is relative to one's experience, understanding and knowledge which is in common with intuition.

e) The next example appears in chapter 17 dealing with quadratic equations. The first special type involve reciprocals, such as,

$$x+\frac{1}{x}=\frac{17}{4}$$

Here it can be observed that $\frac{17}{4}=4+\frac{1}{4}$ and hence either $x=4$ or $x=\frac{1}{4}$

In relation to using Vilokanam, the question is, can this aspect of $\frac{17}{4}=4+\frac{1}{4}$ be easily seen? On first looking at this reciprocal equation an experienced mathematician even at school level will immediately tend to start transposing terms in order to obtain the regular quadratic, $4x^2-17x+4=0$ and then factorise to find, $(4x-1)(x-4)=0$. The indication from the text is to look before you leap. In other words, the reader should spend a little time observing the problem before jumping into any well-practiced or rehearsed methods. This is an important aspect of a good approach when using Vedic techniques.

f) Another example in which good knowledge of arithmetic is required is found in the next reference in relation to solving a special type of biquadratic equation. In particular he gives the example of solving, $(a+1)^4+(a-1)^4=706$.

For the first pair of roots he spots that $706=625+81=5^4+3^4$ from which $a=\pm 4$. This is "by mere inspection"!

For the second pair of roots he expands the brackets, which eliminates odd powers of *x*, and arrives at the quartic, $a^4+6a^2-352=0$. Having obtained $a^2=16$, the other pair can easily be found as $\pm\sqrt{22}$.

The next four references are in chapter 21 dealing with simultaneous quadratic equations.

The equations, $5x-y=17$, $xy=12$.

The first is squared to give, $25x^2 - 10xy + y^2 = 289$, and the second is multiplied by 20 to give $20xy = 240$. These two are then added and factorised to give, $(5x + y)^2 = 529$, from which, $5x + y = \pm 23$. By adding this to the first equation he arrives at $10x = 40$ or -6 and then proceeds to find the two y values of 3 or -20.

He notes that one set of values can be found by Vilokanam and upon inspecting the first two equations one can easily see that $x = 4$ and $y = 3$ do indeed fit.

The next example is similar: $4x - 3y = 7$, $xy = 12$. Here, he uses Vilokanam to find one pair of solutions, namley, $x = 4$ and $y = -3$, the other two solutions are found by a method similar to the previous example.

The other two examples are similar.

Here he assumes knowledge of the two identities,

$x^3 - y^3 = (x^2 + xy + y^2)(x - y)$ and $x^3 + y^3 = (x^2 - xy + y^2)(x + y)$

The next reference is in Chapter 23 on Partial Fractions. In this chapter he describes what is commonly known as the 'cover up' method but which he ascribes to the Paravartya sutra meaning Transpose and Apply.

The very first example is $\frac{2x+3}{(x+1)(x+2)}$

to which he comments, "also available by mere Vilokanam".

The reason for this is that the numerator is the same as the sum of the two binomial denominators.

Therefore $\frac{2x+3}{(x+1)(x+2)} = \frac{1}{x+1} + \frac{1}{x+2}$

The next two references are in Chapter 27 on Straight Division. These are simple applications. Whilst describing the method of division by Vertically and Crosswise, two of the examples can be solved by inspection. These are 888 ÷ 672 and 1111 ÷ 839.

2. Experience, Prior Knowledge and Intuition

On reading through all these references it becomes clear that the use of this sutra depends upon the personal level of experience and prior knowledge. Vilokanam therefore has a strong element of subjectivity inasmuch as it rests upon familiarity developed through previous work in mathematics. This emphasises that the personal approach is an essential feature of Vedic mathematics.

Vilokanam is close to, if not the same as, intuition – the ocurrence of an instaneous realisation. Although there exist natural intuitions that we often call instinct the intuitive realisations in mathematics do depend on past experience.

The 1974 Fields medalist, Enrico Bombieri, described intuition through an anecdotal account involving a mistake.

"One story, perhaps an urban legend, is that a senior expert in real analysis was once sent a paper that "proved" a surprising theorem. The expert looked at the proof, and was immediately skeptical. The "theorem" seemed to be too surprising—his intuition based on his great experience, was that the theorem could not be true. Yet even after hours of studying the proof he could not find any mistakes. But his intuition continued to bother him. He finally looked even more carefully, and found the problem. The author of the proof had used a lemma from a famous topology book. He had used the lemma exactly as it was stated in the famous textbook. But there was a typo in the book. Somehow the words "closed" and "open" had been exchanged in the statement of the lemma. This made the lemma false, caused a gap in the proof of the surprising theorem, and left the poor author with a buggy paper."

Intuitive realisations can ocurr at any level of experience and even, as psychologists concur, in very young children with virtually no experience.

3. Special Cases

In all cases in his book Tirthaji refers to Vilokanam in special cases where he is dealing with a more general case. He lays great emphasis on special cases and this is a strong feature of Vedic mathematics simply because there are easy methods to solve problems that have particular characteristics.

This is corroborated again and again in the text where various special methods are brought to bear on solving a single problem always with a view to finding the path of least action and leaving the reader to choose their own way.

Concerning special cases and general cases, Tirthaji says,

"With regard to every subject dealt with in the Vedic Mathematical Sutras, the rule generally holds good that the sutras have always provided for what may be termed the 'General Case' (by means of simple processes which can be easily and readily – nay, instaneously applied to any and every question which can possibly arise under any particular heading.

"But, at the same time, we often come across special cases which, although classifiable under the general heading in question, yet present certain additional and typical characteristics which render them still easier to solve. And, therefore, special provision is found to have been made for such special cases by means of special Sutras, sub-Sutras, corollaries etc., relating and applicable to those particular types alone.

"And all that the student of these Sutras has to do is to look for the special characteristics in question, recognise the particular type before him (or her) and determine and apply the special formula prescribed therefor.

"And, generally speaking it is only in case no special case is involved, that a general formula has to be resorted to."

This clearly sets out a characteristic of Vedic mathematics where there is a requirement to always look for any special methods that lead to quick easy solutions to problems. And this includes situations where Vilokanam can operate.

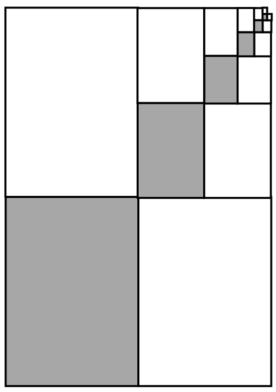
4. Look Before You Leap

People are creatures of habit but also have intelligence that can rise beyond habitual behaviours. In conventional mathematics education students learn blanket methods and algorithms to solve particular problems. Wishing to feel safe, most will stay within their comfort zone and keep with the blanket method they've been taught. However, Vedic mathematics has general case algorithms but encourages dealing with special cases using easier methods. And this greatly assists students to develop and expand their intelligence.

One key leading to this expansion is to encourage students to look closely at a particular problem to find a possible easy solution before reverting to habit - in other words to look before leaping. There are many problems in mathematics that can be solved by 'alternative looking'.

Here is an example.

A rectangle is divided into four as shown and the lower left-hand corner is shaded. The upper right-hand corner is divided into four and its lower left-hand corner is shaded. The process is continued ad infinitum. What fraction of the whole is shaded?



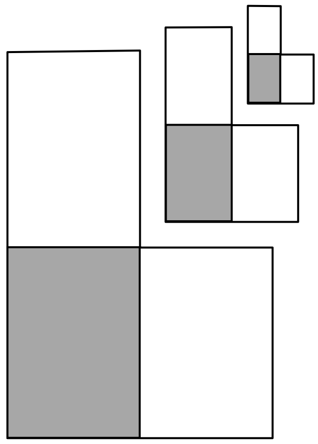
This can be solved by considering the sum of the infinite series,

$$\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+\cdots = \frac{1}{4}+\frac{1}{4^2}+\frac{1}{4^3}+\cdots$$

Using the formula for the sum of an infinite geometric progression,

$$S_\infty = \frac{1/4}{1-1/4} = \frac{1}{3}$$

Another way of solving this is to look at the problem in a different way and once this is done the solution is found by mere observation. The diagram is split up into a series of "L" shapes as shown below.



The figure above shows a method that requires no calculation but just observation.

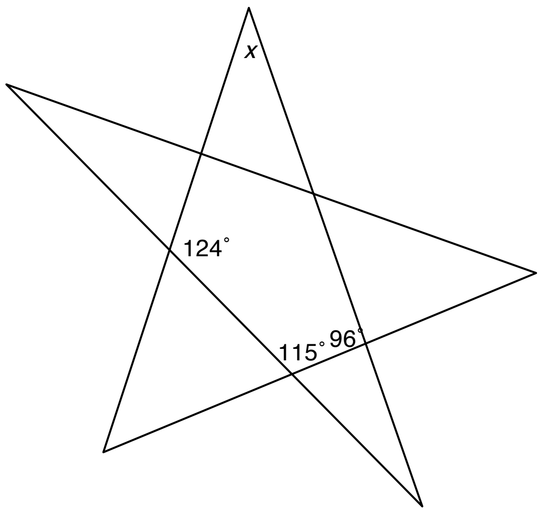
Each “L” shape has one third shaded. There are an infinite number of “L” shapes and so one third of the whole is shaded.

Looking for alternative methods to solve problems obviously leads to the development of strategic thinking. Such development of strategic thinking skills is a prime feature of Vedic mathematics. It is also a characteristic of mathematics graduates, which in turn leads to high levels of employability.

5. By Mere Observation

There are many problems in mathematics that require good observation that are then followed by some calculation. The sutra can be seen to work in such cases as well but only for part of the problem.

In the example below, it is easy to launch into finding adjacent angles to those given in the attempt of chasing down the missing angle. However, this turns out to be a fruitless exercise. The solution is found by looking at the quadrilateral containing the three known angles and *x*.



Once this quadrilateral is seen the answer arrives very easily through a simple calculation.

But how does this recognition or "seeing" take place? It comes about by having a still mind when looking at the problem, of not jumping to the habitual thought processes but by looking afresh.

Here is an anecdotal example of how lack of observation leads to mistakes. Recently, I was working through an A level problem involving an exponential function concerned with the population of mice, *N*, on an island, *t* hours after a starting time. The function was $N = 3 + 7e^{-0.25t}$.

One part of the problem was to derive the rate of change of N, with resepct to time, as a function of N as,

$$\frac{dN}{dt} = \frac{N(300 - N)}{1200}$$

The next part of the problem read, "find the the number of hours it takes for the rate of change to be a maximum."

On seeing the word "maximum" I immediately thought to solve $\frac{dN}{dt} = 0$, ie $N = 300$ and then to find *t* at this value. But I kept getting a nonsense answer and although knew something was wrong I just could not see the mistake. Eventually, I asked my brightest student for help! He looked at it and simply pointed out that the question was asking for the maximum rate of change and not the maximum value of *N*. The second derivative was required to be equated to zero.

It turned out to be a useful teaching point. The other students could then see how important it is to read the question very carefully – to examine every detail. And this is all part of the power of observation. Had I read the question with greater care the mathematics required would have been immediately available.

6. Discovering Pattern

Leading on from this idea of 'look before you leap' is the discovery of pattern.

Sometimes observation of a potential pattern will lead to questions and further investigation.

The square root of 2 is a good example. In decimal form, $\sqrt{2} = 1.4142135...$ and taking the digits in pairs a certain pattern becomes obvious, 14 14 21 35, all of which are in the 7 times table. The question naturally arises as to whether this is a coincidence or not and does the pattern continue in the same way? So this is a case where observation leads to an intuitive sense that there must be something there and this in turn leads to investigating the mathematics involved.

$$\sqrt{2} = \frac{7\sqrt{2}}{7} = \frac{7 \times 2}{7\sqrt{2}} = \frac{14}{\sqrt{98}} = 14(100 - 2)^{-\frac{1}{2}}$$

The binomial expansion of this soon reveals why the pattern starts with multiples of 7.

This shows that an initial intuitive sense of pattern is supported by logic. The digits of √2 are not random but highly ordered.

When working in mathematics both rigour and intuition are necessary. Fields medal winner, Terence Tao, described the relationship between rigour and intuition.

"The point of rigour is not *to destroy all intuition; instead, it should be used to destroy* bad *intuition while clarifying and elevating* good *intuition. It is only with a combination of both rigorous formalism and good intuition that one can tackle complex mathematical problems; one needs the former to correctly deal with the fine details, and the latter to correctly deal with the big picture."*

An interesting aspect of Tao's description is his reference to 'finer details' and 'big picture'. Mathematical rigour requires analysis, whilst intuition relates to the whole and so uses synthesis. And this also relates to use of the right and left sides of the brain in which the left side is more active when performing analysis and the right side is more active when recognising wholes such as when seeing pattern.

Conclusion

The Vilokanam sutra is central to Vedic mathematics and, in relation not only to other sutras but also to conventional mathematical rules, principles and algorithms reveals a different dimension – a dimension that is subjective, less mechanical and more conscious. It tells us that performing mathematics is personal as well as universal – personal because of the dependence on past experience and the immediacy of knowledge and universal because everybody experiences observational solutions to problems.

Tirthaji places visualisation as a causal feature of Vedic mathematics as is made clear in his Prolegomena.

"Immemorial tradition has it and historical research confirms the orthodox belief that Sages, Seers and Saints of ancient India (who are credited with having observed, studied and meditated in the Aranya (i.e. in forest solitude) – on physical Nature around them and deduced their grand Vedantic Philosophy therefrom as the result not only of their theoretical reasonings but also of what may be more fittingly described as Trua Realisation by means of actual VISUALISATION) seem to have similarly observed, studied and meditated on the mysterious workings of numbers, figures etc., of the mathematical world (to wit, Nature) around them and deduced their Mathematical Philosophy therefrom by a similar process of what one may, equally correctly, describe as processes of Trua Realisation by means of actual VISUALISATION."

In some respects finding a solution to any mathematical problem involves a touch of Vilokanam. When facing a problem there can be a moment of realisation in which the precise mathematical knowledge that will solve the problem is revealed to the observer.

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