BINOMIAL EXPANSIONS AND THE NAUGHTY CALCULUS

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Abstract

When the Maclaurin series for binomial expansions is looked at in terms of calculus a very fast an easy method for expanding binomials for negative or fractional powers is revealed. It also works when the powers are natural integers. The Vedic maths sutras involved are Calanakalanabhyam – Differential Calculus and Ekadhikena Purvena – By one more than the one before. This paper explains the easy formula, particularly useful for calculator-free exam questions and shows the modus operandi through examples of exam questions. In some simple examples involving negative powers the method is compared with algebraic division.

This paper shows that by viewing Maclaurin series from the perspective of differentiation and integration there is an easily remembered formula for binomial expansions for powers that are positive or negative integers or for fractional powers. The algebraic processes of differentiation and integration of single terms are very simple. Although they would not understand the meaning, even 12 year-olds can perform the algebraic manipulations.

I use the term "Naughty Calculus" because numbers will be treated as algebraic terms that can be differentiated or integrated. For example, the derivative of 5^3 will be taken as 3×5^2, etc.

The formula sets down the product of the two terms of the binomial, the first raised to the power of *n* and the second to the power of zero. This provides the first term of the expansion. The second term is produced by differentiating the first term and integrating the second term. Successive terms are found by repeating the process of differentiation and integration.

This process simultaneously creates the correct powers of the two terms and also the binomial coefficients.

The Maclaurin series is,

$$f(x) \;=\; f(0)+f'(0)x+f''(0).\frac{x^2}{2}+f'''(0).\frac{x^3}{2.3}+\cdots$$

Each term can be seen as the product of a derivative and an integral.

In the case of a binomial raised to some power, *n*, say, $(a+x)^n$, successive derivatives are,

$$n(a+x)^{n-1},\; n(n-1)(a+x)^{n-2},\; n(n-1)(n-2)(a+x)^{n-3}, \text{ etc.}$$

Evaluating each of these for $x=0$ gives, na^{n-1}, $n(n-1)a^{n-2}$, $n(n-1)(n-2)a^{n-3}$, and so on.

The terms, $x,\; \frac{x^2}{2},\; \frac{x^3}{2.3},\; \frac{x^4}{2.3.4},$ etc., are successive integrals starting with x^0.

From this we can generalise the binomial expansion formula as,

$$(a+b)^n = \sum_{r=0}^{n} D_r(a^n).I_r(b^0)$$

where, $D_r(a^n)$ is the rth derivative of a^n with respect to a and $I_r(b^0)$ is the rth integral of b^0 with respect to b. Note that neither a nor b have to be algebraic. However, if they are numbers they should be treated as algebraic entities.

***n* as a Positive Integer**

Example 1 Expand $(3+x)^4$

Set down the product of the first term raised to the power of 4 and the second term raised to the power of 0.

$$3^4.x^0$$

To obtain the second term, differentiate 3^4 (with respect to 3) to give 4.3^3 and integrate the second term to give x^1.

We then have,

$$3^4.x^0 \; + \; 4.3^3.x^1$$

For the third term, differentiate 4.3^3 to give $3.4.3^2$ and integrate x^1 to give $\frac{x^2}{2}$.

By continuing in like manner the expansion becomes,

$$3^4.x^0 \; + \; 4.3^3.x^1 \; + \; 3.4.3^2.\frac{x^2}{2} \; + \; 2.3.4.3^1.\frac{x^3}{2.3} \; + \; 1.2.3.4.3^0.\frac{x^4}{2.3.4}$$

and on simplifying find,

$$(3+x)^4 = 81 \; + \; 108x \; + \; 54x^2 \; + \; 12x^3 \; + \; x^4 .$$

Now although this may seem a little complicated for simple expansions when n is a positive integer the method does come into its own for negative and fractional powers.

***n* as a Negative Integer**

Example 2 Expand $(3-2x)^{-2}$ in ascending powers of x up to the term in x^3.

In this case, $a=3$ and $b=(-2x)$.

$$\begin{aligned}(3-2x)^{-2} &= 3^{-2}.(-2x)^0 \; + \; -2.3^{-3}.(-2x)^1 \; + \; -3.-2.3^{-4}.\frac{(-2x)^2}{2} \; + \; -4.-3.-2.3^{-5}.\frac{(-2x)^3}{2.3} + \cdots \\ &= \tfrac{1}{9} \; + \; \tfrac{4}{27}x \; + \; \tfrac{4}{27}x^2 \; + \; \tfrac{32}{243}x^3 + \cdots\end{aligned}$$

Once the intitial term is set down the first factor is differentiated and the second factor is integrated to obtain the second term. The processes of differentiation and inttegration then continue as far as is requred.

In regular text books students are told to take out a factor of 3 first giving $\left\{3^{-2}(1-\tfrac{2}{3}x)^{-2}\right\}$ then to expand the binomial and, finally, to then multiply each term by 3^{-2}. But this method is more direct and requires no such algebraic manipulation. The reason for taking out a factor of 3 is so that the expansion adheres to the conventional formula,

$$(1+ax)^n = 1 + nax + n(n-1)(ax) + n(n-1)(n-2)\frac{(ax)^2}{2!} + \cdots$$

but there is no need for this when using the calculus formula.

When *n* is a Fraction

Example 3 Expand $(2+x)^{\frac{1}{2}}$ in ascending powers of *x* up to the term in x^2.

$$\begin{aligned}(2+x)^{\frac{1}{2}} &= 2^{\frac{1}{2}}x^0 + \tfrac{1}{2}.2^{-\frac{1}{2}}.x^1 + -\tfrac{1}{2}.\tfrac{1}{2}.2^{-\frac{3}{2}}.\frac{x^2}{2} + \cdots \\ &= \sqrt{2} + \tfrac{\sqrt{2}}{4}x + \tfrac{\sqrt{2}}{32}x^2 + \cdots\end{aligned}$$

One main advantage with this calculus method is that the formula is very easy to remember. Once the first term is established it is just a case of finding successive derivatives and integrals and then simplifying the terms.

Radius of Convergence

Of course, for negative and fractional powers in conventional mathematics, the radius of convergence, or limit of convergence, must also be taken into account.

For $(a+bx)^n$,

$$(a+bx)^n = a^n(1+\tfrac{b}{a}x)^n$$

and this is valid for $-1 \le \tfrac{b}{a}x \le 1$ or $-\tfrac{a}{b} \le x \le \tfrac{a}{b}$

Conclusion

The relationship between differential and integral calculus and the binomial expansion formula has not hitherto been utilised for school students. This is a pity as the formula is so much easier to remember than that of the Maclauren series.