EVALUATION OF PERMUTATIONS AND COMBINATIONS

Kenneth Williams, Vedic Mathematics Academy, U.K.

Abstract

Evaluating permutations and combinations involves finding the product of a sequence of whole numbers. This paper looks at the Vedic techniques that may be applied in finding such strings of products. We find two Vedic devices that are particularly useful (base multiplication and the use of the so-called 'special' numbers), and a special case. We also consider the case where values are large and need to be approximated.

1. INTRODUCTION

A *permutation* is an arrangement of objects taken from a set of objects where the order of the arrangement is significant. For example, if we take two letters from the three letters *A*, *B*, *C*, for example, we find there are 6 possible permutations: *AB*, *AC*, *BA*, *BC*, *CA*, and *CB*.

The notation for the number of ways of arranging *r* objects from a set of *n* objects is written ${}_nP_r$ and

$${}_nP_r = \frac{n!}{(n-r)!} = n(n-1)(n-2) \ldots (n-r+1).$$

[*n*!, pronounced "*n* factorial", means the product of all the whole numbers from *n* down to 1.]

In the above example ${}_3P_2 = \frac{3!}{(3-2)!} = 6$.

A *combination* is when the order of the selected objects is *not* significant. For example, if we select two letters from the three letters *A*, *B*, *C* we find there are 3 possible selections: *AB*, *AC*, *BC* (since *AB* is the same as *BA* etc.).

$${}_nC_r = \frac{n!}{r!(n-r)!} = n(n-1)(n-2) \ldots (n-r+1)/r!$$

In the above example ${}_3C_2 = \frac{3!}{2!(3-2)!} = 3$.

In both cases the denominators of the fractions above always cancel out completely with the numerators (as they must do since they give a number of arrangements or selections, which must have whole number values) and so we find we are dealing with the *product of a series of positive whole numbers*. In the case of permutations we have a series of *consecutive* numbers.

There are many approaches to evaluating these products depending on the properties of the numbers involved.

We may note:

1) a permutation ${}_nP_r$ is a product of r numbers from n downwards;

2) we can switch from a permutation to a combination by dividing the corresponding permutation by r!

3) since ${}_nC_r = {}_nC_{n-r}$ if we had ${}_{19}C_{12}$ for example we would not want to consider the product of 12 numbers from 19 downwards divided by 12!, but would convert ${}_{19}C_{12}$ to ${}_{19}C_7$.

Permutations and combinations are important as they arise in a great many practical situations.

Of course electronic calculating devices can be used for evaluating them, but our interest here is in the Vedic methods that can be applied.

2. VEDIC DEVICES USED

Having several numbers to multiply means we can ourselves decide the sequence of combination, and so we can take advantage of various special Vedic devices.

2.1. Base Multiplication (Sutra: *All from 9, Last from 10)*)

This method is now well-known. For example 97 × 96 = 93/12. We see the given numbers are close to a base of 100. The first part of the answer is one of the numbers reduced by the deficiency of the other number from the base (97 – 4 = 93 or 96 – 3 = 93), and the second part is the product of the deficiencies (3×4 = 12).

Proof: $(x + a)(x + b) = x(x + a + b) + ab$,
where x is a power of 10 and a and b are deviations from that.

Example A
${}_{97}P_2$ means the product of 2 numbers from 97 downwards, so ${}_{97}P_2 = 97 \times 96 = 9312$, using base multiplication.

Example B
${}_{49}C_3$ is a combination, so ${}_{49}C_3 = \dfrac{49\times48\times47}{6}$ i.e. the product of 3 numbers from 49 downwards divided by 3! We can cancel the 6 with the 48 and apply two of the remaining factors of 48 to double 49 and 47, thereby bringing the products close to 100. So ${}_{49}C_3 = 98\times94\times2 = 9212\times2 = 18424$.

That is, we can organise the calculation in such a way as to use base multiplication.

Similarly, when numbers are close to a multiple (or sub-multiple) of a power of 10 we may use: $(nx + a)(nx + b) = nx(nx + a + b) + ab$.

Example C
${}_{204}P_2 = 204 \times 203 = 2{\times}207\ /\ 4{\times}3 = 41412$.

This method is extendible to products of more numbers. For example 91×96×99 or 84×91×96×103.

Example D
${}_{13}P_7 = 13 \times 12 \times 11 \times 10 \times 9 \times 8 \times 7$.

We could simply start at the left and multiply the numbers, but with a little thought we can pair the numbers in a useful way: (13×7) × (12×8) × (11×9) × 10.

That is 91×96×99×10.

We find the product of the first three numbers using a base of 100:

$$\begin{array}{lll} 91 & - & 9 \\ 96 & - & 4 \\ \underline{99} & \underline{-} & \underline{1} \\ \multicolumn{3}{l}{\underline{86\,/\,49\,/\,\overline{36}}} \end{array} = 864864.$$

Proof: $(x + a)(x + b)(x + c) = x^2(x + a + b + c) + x(ab + bc + ca) + abc$.[2]

So ${}_{13}P_7 = 8648640$.

Similarly, with a product of three numbers close to a *multiple* of a power of 10 we may use: $(nx + a)(nx + b)(nx + c) = n^2x^2(nx + a + b + c) + nx(ab + bc + ca) + abc$.

2.1.1 Approximations

This method is useful too when the numbers involved are large and only an approximation is required. This is because we can get the various sections of the answer from left to right, stopping when the desired accuracy is achieved. And the *number* of sections (which equals the number of numbers being multiplied) tells us the order of magnitude of the product.

So in the 91×96×99 example above (in which the answer comes in three sections) we begin by selecting one of the numbers and taking the other two deficiencies from it: we get 86. And since our base is 100, and there are two more sections after the first one we will need to place 4 zeros after the 86. So 91×96×99 ≈ 860,000 and ${}_{13}P_7 \approx 8600{,}000$.

2.2. Special numbers (Sutra: *By Mere Observation*)

There are certain numbers which have the property that they have *useful factors* and are also *easy to multiply with*. For example 1001 = 7×11×13.

1001 is very easy to multiply by. For example:
1001×346 = 346346,

1001×34 = 1001×034 = 34034,
1001×3456 = 3456_3456 = 3459456.

So any product that contains the factors 7, 11 and 13 can be found using this special number, 1001.

There are a great many of these special numbers. We can list a few:

2 × 5 = 10
67 × 3 = 201
43 × 7 = 301
17 × 53 = 901
7 × 11 × 13 = 1001
23 × 29 × 3 = 200l
17 × 6 = 102
13 × 8 = 104
29 × 7 = 203
19 × 16 = 304
2^{10} = 1024

31 × 13 = 403
23 × 22 = 506
37 × 3 = 111
37 × 3 × 7 × 13 = 10101
11 × 9 = $10\bar{1}$
27 × 37 = $100\bar{1}$
23 × 13 = $30\bar{1}$
17 × 47 = $80\bar{1}$
3 × 31 × 43 = $400\bar{1}$

Example E
$_{13}P_7$ is Example D again.

13 × 12 × 11 × 10 × 9 × 8 × 7 does contain 7×11×13 and so the product can be written as: 12×9×8×1001×10.

12×9×8 is found to be 864, so $_{13}P_7$ = 8648640.

Example F
$_{26}C_4 = \frac{26\times25\times24\times23}{24}$. Here, after cancelling the 24 we note we can use 13×23 = $30\bar{1}$. So $_{26}C_4 = 30\bar{1} \times 50 = 150\bar{5}\bar{0} = 14950$.

2.3. First Figures the Same, Last totalling 10 (Sutra: *By One More than the One Before*)

This is where we are multiplying two numbers where the first figures are the same and the last add up to 10.

For example, 62 × 68.

We multiply the first figure by the number ‘one more’: 6×7 = 42.
And we multiply the last figures: 2×8 = 16.
So 62 × 68 = 42/16 = 4216.

Example G
$_{46}P_3$ = 46×45×44
= 2024 × 45

= 1012 × 90
= 91080.

This method is well-known in the Vedic system[1,2] and we will use it for the special case in Section 4.

2.4 We will also be using the *vinculum*. This is a bar placed over a digit to make its value negative, and in fact has featured already in Sections 2.1 and 2.2.

3. FURTHER EXAMPLES

Example H
${}_{37}P_3$ = 37 × 36 × 35 = 111×105 × 4 = 11655 × 4 = 46620.

Here we can obtain two numbers close to 100 by coupling 37 with 3 and 35 with 3, and doubling twice.

Example I
${}_{8}P_5$ = 8 × 7 × 6 × 5 × 4.

Of course there are many ways to find this and we can choose the one that appeals to us most.

Seeing the presence of 5, it makes sense to pair it with an even number. We may also recall the special number 301 = 43 × 7. And since the first three numbers give 48×7 we can find this as 301 + 5×7 = 336 (since 48×7 = 43×7 + 5×7).
Now we only have to multiply by 20 to get ${}_{8}P_5$ = 336 × 20 = 6720.

This example illustrates that we can use 301 = 43×7 even when the number 43 is not present.

Example J
${}_{18}P_5$ = 18 × 17 × 16 × 15 × 14.

The base of 100 can be used here *by looking carefully at the factors of the numbers*.

We can take 17×6 = 102 and 15×7 = 105.
That leaves 16×6 = 96: also close to 100.

The multiplication then gives $103 / \overline{18} / \overline{40}$ = 1028160 (by the base method of section 2.1).

Example K

$${}_{19}C_7 = \frac{19\times18\times17\times16\times15\times14\times13}{7\times6\times5\times4\times3\times2}.$$

We now proceed to cancel.
Note that we can cancel symmetrical products in the denominator in this case. That is 7×2

cancels with the 14 and 6×3 cancels with the 18.
The remaining 5×4 can cancel with the 15 and 16 giving:

19 × 17 × 4 × 3 × 13 as the required value.

We can now look for special numbers. If we are familiar with 19×17×13 = $420\bar{1}$ we can simply multiply this by 12 to get the answer 50388.

Alternatively, by counting 4×3 as 12 and careful pairing of the four numbers 19, 17, 12, 13 we can find (19×13) × (17×12) = 247 × 204.
And we can multiply these numbers using a base of 200 to get 50388 again.

4. A special case: ${}_{5n-1}P_4$, $n \geq 1$ (Sutra: *By One More than the One Before*)

Especially easy is the case where we have four numbers to multiply which run down from a number that is 1 below a multiple of 5.

. . . 20 19 18 17 16 15 14 13 12 11 10 . . .

These blocks of 4 numbers are almost consecutive, the numbers not included being ones that are easy to multiply by (10, 15, 20 etc.).

Example L
${}_{14}P_4$. We note that 14 is 1 below 15 (a multiple of 5) and we have a product of 4 numbers, so this special case applies.
This is 14×13×12×11, and the outer and inner products give 154 and 156.

Here we notice we can use Vedic method of section 2.3 that applies when the first figures are the same (15 in this case) and the last figures add up to 10 (4+6=10 in this case).

We get 15×16 / 4×6 = 240/24.
So ${}_{14}P_4$ = 24024.

This special method will always be applicable for permutations of the type ${}_{5n-1}P_4$.
Such products always end in 24.

Example M
${}_{19}P_4$ = 19×18×17×16 = 304 × 306 = 30×31 / 4×6 = 93024.
Here we find 19×16 = 304 using the Base method or Proportion.
We hardly need to find the other (inner) product because we know that this special case applies. We simply find 30×31 and place 24 after it.

This special method can be used in other situations.

Example N

${}_{20}P_5$ = 20 × 19 × 18 × 17 × 16, we notice the sequence 19×18×17×16 from the above example contained in it, and so ${}_{20}P_5$ = 20 × 93024 = 1860480.

Example O

$${}_{36}C_7 = \frac{36\times35\times34\times33\times32\times31\times30}{7\times6\times5\times4\times3\times2}.$$

We notice the special case can be applied to 34×33×32×31.
So we can try to restrict our cancelling to the other numbers. That is, 7×5 cancels with the 35, and 6×3×2 cancels with the 36.

This leaves ${}_{36}C_7 = 34\times33\times32\times31 \times \frac{30}{4} = 1113024 \times \frac{30}{4} = 8347680$.

Example P

${}_{26}P_6$ = 26 × 25 × 24 × 23 × 22 × 21.

There are several choices:
a) We may note the sequence 24×23×22×21 as described in the above special case.
b) We may use the contained 26×4 = 104 and 25×4 = 100.
c) Or use 13×23 = $30\bar{1}$ and 25×4 = 100.
d) Or 2×22×23 = 1012 and 25×4 = 100.

e) But best is possibly to use the contained 7×11×13 = 1001 with 25×4 = 100.
That leads to finding 23×9 = 207, which when multiplied by 1001 gives 207207, which we can then multiply easily by 8 to get 1657656.
So ${}_{26}P_6$ = 165765600.

4.1 Proof

We need to show that ${}_{5n-1}P_4$ leads to the product of two numbers in which the last digits add up to 10 and the other figures are the same.

$${}_{5n-1}P_4 = (5n - 1)(5n - 2)(5n - 3)(5n - 4) = (25n^2 - 25n + 4)(25n^2 - 25n + 6). \qquad [1]$$

But $25n^2 - 25n = 25n(n - 1)$ and since $n(n - 1)$ is the product of two consecutive numbers, one of them must even, making $25n^2 - 25n$ a multiple of 10 (and hence not affecting the units digits in [1]).

It follows that $(25n^2 - 25n + 4)(25n^2 - 25n + 6)$ is the product of two numbers ending in 4 and in 6 respectively and that the other digits will be the same.

5. APPROXIMATIONS FOR LARGE VALUES (Sutra: *Special and General*)

The values of these permutations and combinations can get very big and in such situations we may require only an approximate answer: perhaps the first digit and the power of 10 attached to it.

Focussing on a central value provides a way to get approximate evaluations without a great deal of work.

Example Q

${}_{25}P_{11}$ is the product of 11 numbers from 25 downwards.

That is: 25 × . . . . × 15. It has a value of about 1.8×10^{14}, so it is a large number.

With 20 as the average of 25 and 15 we may approximate ${}_{25}P_{11}$ with 20^{11}.

Recalling the special number $2^{10} = 1024 \approx 10^3$ we may proceed as follows.

${}_{25}P_{11} \approx 20^{11} = 2^{11}\times10^{11} = 2048\times10^{11} \approx 2\times10^{14}$.

Example R

${}_{19}P_6 = 19 \times 18 \times 17 \times 16 \times 15 \times 14$.

For a quick approximate value we can use the fact that the 16 roughly in the centre of these six numbers is 2^4.

So we consider $(2^4)^6$ (since there 6 numbers).

This is $2^{24} = (2^{10})^2 \times 2^4 \approx 16 \times 10^6$.
(Note ${}_{19}P_6 = 19{,}535{,}040$)

Example S

${}_{22}P_7 = 22 \times 21 \times 20 \times 19 \times 18 \times 17 \times 16$.

19 being in the centre we consider 19^7.
But since 20 is close we may use that easier number and find $20^7 = 128 \times 10^7$.
So we can approximate ${}_{22}P_7$ with 1×10^9.
(Note ${}_{22}P_7 = 859{,}541{,}760$)

This does raise the question of the difference between 20^7 and 19^7. And we can obtain this by considering $(n + a)^p$, where n is a base number (20 in this example), a is a small increase or decrease in n (a = -1 here) and p is the power involved.

Now expanding and considering the first two terms we find $(n + a)^p \approx n^{p-1}(n + pa)$

So we may approximate 19^7 with $20^6(20 - 7\times1) = 64{,}000{,}000 \times 13 \approx 8.3 \times 10^8$.

Example T

${}_{36}C_7$. To get an approximation for this earlier example we can cancel the 36 and 35 as before.

This leaves ${}_{36}C_7 = \dfrac{34\times33\times32\times31\times30}{4}$.

We may approximate this with $\frac{32^5}{4} = \frac{32}{4} \times 2^{20} = 8 \times 10^6$.

6. SUMMARY

Permutations and combinations can be evaluated without resorting to a calculator. The Vedic system offers two devices that are especially useful for handling products of several numbers: base multiplication and the special numbers. The special case where we need a product of four consecutive numbers running downwards from a number of the form $5n–1$ is also very useful.

In addition to finding exact values we can also get approximate values by using an average.

7. CONCLUDING REMARKS

No doubt there are other Vedic methods that can be used in this area, and maybe simpler devices than those shown here. There are probably other 'special cases' that can be applied.

Since there is such a variety of options available to find any given product of numbers it would be educationally beneficial for students to find their own creative way to obtain them.

References

[1] Bharati Krishna Tirthaji Maharaja, (1965). Vedic Mathematics. Delhi: Motilal Banarasidas,.

[2] Williams. K. R. (2017). Discover Vedic Mathematics. U.K.: Inspiration Books.