FACTORISATION AND DIFFERENTIAL CALCULUS

Kenneth Williams, Vedic Mathematics Academy, U.K.

Abstract

Chapter 22 of Sri Bharati Krishna Tirthaji's book "Vedic Mathematics" is titled "Factorisation and Differential Calculus" and shows two interesting connections between the factors of polynomials and differential calculus. Tirthaji also gives some hints about other applications in this area.

Unlike most of the chapters in the book this one is rather brief and there is a case for explaining the content further, with more examples, and also developing the other applications mentioned. This paper is an attempt to approach the chapter in this way.

1. INTRODUCTION

First, the paper will explain the two main methods shown in Chapter 22[1], then some ideas will be given about the other comments made by Tirthaji regarding Leibniz theorem, Maclaurin's theorem and Taylor's theorem.

2. FACTORS AND DIFFERENTIALS OF POLYNOMIALS

The first three examples Tirthaji gives to illustrate this connection are as follows:

"(1) $x^2 + 3x + 2 = \overset{a}{(x + 1)}\overset{b}{(x + 2)}$

$\therefore D_1$ (the first differential) $= 2x + 3 = (x + 1) + (x + 2) = \Sigma a$

(2) $x^3 + 6x^2 + 11x + 6 = \overset{a}{(x + 1)}\overset{b}{(x + 2)}\overset{c}{(x + 3)}$

$\therefore D_1 = 3x^2 + 12x + 11 = (x^2 + 3x + 2) + (x^2 + 5x + 6) + (x^2 + 4x + 3)$

$= ab + bc + ac = \Sigma ab.$

$D_2 = 6x + 12 = 2(3x + 6) = 2[(x + 1) + (x + 2) + (x + 3)]$

$= 2(a + b + c) = 2\Sigma a = 2!\Sigma a.$

(3) $x^4 + 10x^3 + 35x^2 + 50x + 24 = (x + 1)(x + 2)(x + 3)(x + 4)$

$\therefore D_1 = 4x^3 + 30x^2 + 70x + 50 = \Sigma abc$

$D_2 = 12x^2 + 60x + 70 = 2\Sigma ab = 2!\Sigma ab$

$D_3 = 24x + 60 = 6(4x + 10) = 3!\Sigma a.$"

After two further but similar examples he writes: "These examples will suffice to show the internal relationship subsisting between the factors of a Polynomial and the successive differentials of that Polynomial; and to show how easily, on knowing the former, we can derive the latter and vice versa."

This is as far as Tirthaji goes and we are left wondering how this may be used or taken further.

Of course the successive differentials of a polynomial are easily found by differentiation term by term. What Tirthaji is pointing out is the connection between the *factors* and the differentials. The following examples show how Tirthaji's results may be applied.

2.1 Example A
Given $f(x) = (x + 2)(x + 5)(x - 3)$ find D_1 and D_2 (i.e. the first two differentials of $f(x)$).

We do not multiply out the brackets and differentiate, which is the usual approach.

Since we have a cubic we can refer to Example 2 above: $D_1 = \Sigma ab$ = the sum of the products of the factors taken 2 at a time $= (x + 2)(x + 5) + (x + 2)(x - 3) + (x + 5)(x - 3)$,

(which may be simplified to give $3x^2 + 8x - 11$).

Similarly for D_2 we have $D_2 = 2!\Sigma a = 2[(x + 2) + (x + 5) + (x - 3)]$
(which again may be simplified to $6x + 8$).

This method avoids the necessity of multiplying out the factors of $f(x)$. And for D_2 especially it is a big saving of time and effort.

2.2 Example B
Given $f(x) = (x + \alpha)(x + \beta)(x + \gamma)(x + \delta)$ find D_3.

From Example 3 above we can immediately say that

$$
\begin{aligned}
D_3 &= 3!\Sigma a \\
&= 6[(x + \alpha) + (x + \beta) + (x + \gamma) + (x + \delta)] \\
&= 6(4x + \alpha + \beta + \gamma + \delta)
\end{aligned}
$$

Again we can get the result immediately without multiplying out the given quartic and differentiating three times.

2.3 Example C
The roots of a cubic are α, β and γ. Find D_2.

From Example 2 we have

$$
\begin{aligned}
D_2 &= 2!\Sigma a = 2[(x - \alpha) + (x - \beta) + (x - \gamma)] \\
&= 2(3x - \alpha - \beta - \gamma)
\end{aligned}
$$

3. DETECTION OF REPEATED FACTORS

In the second part of Chapter 22 Tirthaji shows how to detect when a polynomial has repeated factors.

The mathematical result which he uses, but does not explicitly state or prove, is that if a polynomial has a repeated factor, then that factor will also be a factor of the differential of the polynomial.

3.1 We can prove this as follows.

Let E be a polynomial of the form $f^n g(x)$ where f is a linear factor of E n times and $n > 1$. Here f is a factor of E which is repeated. So if $n = 2$ then E will have f^2 as a factor.

Now differentiating E by the product rule:
$E' = n f^{n-1} f'(x)\, g(x) + f^n g'(x)$
$\quad = f^{n-1}[n f'(x)\, g(x) + f g'(x)]$

showing that f is a factor of E'.

An implication or corollary of this is that if $n = 3$ then f will be a factor not only of the 1st differential but of the 2nd differential too.

And so on for $n = 4, 5, 6$ etc. That is, f will be a factor of all differentials of E down to the (n-1)th differential.

3.2.1 Tirthaji's first example is as follows.

"(1) Factorise $x^3 - 4x^2 + 5x - 2$

$\therefore \dfrac{dy}{dx} = 3x^2 - 8x + 5 = (x - 1)(3x - 5)$

Judging from the first and the last coefficients of E (the given expression), we can rule out $(3x - 5)$ and keep our eyes on $(x - 1)$.

$\therefore\ D_2 = 6x - 8 = 2(3x - 4)\ \therefore$ we have $(x - 1)^2$

$\therefore$ (According to the *Ādyam Ādyena* Sūtra $E = (x - 1)^2(x - 2)$."

3.2.2 Explanation of Tirthaij's Example 1

Having differentiated the given cubic we try to factorise $\dfrac{dy}{dx}$ and any factor thus found *is a contender* to be a repeated factor.

So when Tirthaji says "we can rule out $(3x - 5)$" he is saying that $(3x - 5)$ cannot be a repeated factor because it cannot be a factor of E.

However the other factor of $\dfrac{dy}{dx}$, i.e. $(x - 1)$, *could be* a factor of E, and by substituting $x = 1$ in E we find that it is indeed a factor, and it must therefore be a repeated factor.

We still do not know at this stage how many times $(x - 1)$ is a factor of E and so we need to examine the 2nd differential.*

That 2nd differential has only $(3x - 4)$ as a factor and so we know that $(x - 1)^2$ is a factor of E and not $(x - 1)^3$.

Having found that $(x - 1)^2$ is a factor of E the remaining factor is easily obtainable using *The First By the First and the Last By the Last* as Tirthaji says.

*It is true that we don't really need to consider the 2nd differential in this case since the last term of E would need to be 1 if $(x-1)^3$ was a factor of E, but the example illustrates the method.

3.3.1 This leads to a neat process in which we successively differentiate a given polynomial looking for contenders for repeated factors, and when we get one we then go back through the differentials to see if those contenders are factors of the earlier differentials. We can see this in Tirthaji's example 4.

"(4) Factorise $2x^4 - 23x^3 + 84x^2 - 80x - 64$

$\therefore\ D_1 = 8x^3 - 69x^2 + 168x - 80$ 1

$\therefore\ D_2 = 24x^2 - 138x + 168 = 6(4x^2 - 23x + 28) = 6(x-4)(4x-7)$ 2

$\therefore\ D_3 = 48x - 138 = 6(8x - 23)$ 3

$\therefore\ D_2 = 6(x-4)(4x-7)$ 4

$\therefore\ D_1 = (x-4)^2(8x-5)$. 5

$\therefore\ E = (x-4)^3(2x+1)$." 6

3.3.2 Explanation of Tirthaij's Example 4

The logic seems to be as follows.

Line 1: Get D_1. We get a cubic and do not wish at this stage to attempt factorisation.
Line 2: Get D_2. We get a quadratic, and these are easy to factorise: we get the factors $(x-4)$ and $(4x-7)$.
Line 3: Get D_3. We are looking to see if $(x-4)$ or $(4x-7)$ can be factors of D_3. We get the $(8x-23)$, so the answer is no.
Line 4: We therefore go back to D_2, and we ask if $(x-4)$ or $(4x-7)$ may be factors of D_1.
Line 5: We note that $(4x-7)$ cannot be a factor of D_1 but $(x-4)$ can. And we verify this with the factor theorem.
Line 6: Now we are asking if $(x-4)$ is a factor of E, and the factor theorem shows that it is, and so $(x-4)^3$ must be a factor of E. We use the *Ādyam* Sūtra to get the full factorisation.

We may note that Tirthaji has included some work that is actually unnecessary. For example, the $(4x-7)$ factor is immediately rejected mentally. Tirthaji includes it for the sake of clarity.

4. TIRTHAJI'S OTHER COMMENTS IN CHAPTER 22

4.1 Leibniz Theorem

The *Guṇaka-Samuccaya* Sutra, which Tirthaji emphasises in this chapter, translates as *All the Multipliers*. Here we may understand it as *All the Factors*. He writes, in the introductory part of the chapter:

"It need hardly be pointed out that the well-known rule of differentiation of a product (i.e. that if $y = uv$, when u and v be the function of x, $\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$) and the *Guṇaka-Samuccaya* Sūtras denote, connote and imply the same mathematical truth."

This again shows a connection between factors and differential calculus.

4.1.1 For example, to find the 1st differential of $x^5 \times e^{2x}$ we may write the two factors, x^5 and e^{2x}, one vertically beneath the other, and put their derivatives next to them[2]:

$$\begin{array}{ll} x^5 & 5x^4 \\ e^{2x} & 2e^{2x} \end{array}$$

Then by cross-multiplication and addition $D_1 = x^5 \times 2e^{2x} + 5x^4 \times e^{2x}$.

4.1.2 For the second derivative we put the first and second derivatives and multiply vertically and crosswise:

$$\begin{array}{lll} x^5 & 5x^4 & 20x^3 \\ e^{2x} & 2e^{2x} & 4e^{2x} \end{array}$$

Then $D_2 = x^5 \times 4e^{2x} + 2 \times 5x^4 \times 2e^{2x} + 20x^3 \times e^{2x}$.

The extra factor, 2, that comes into the middle term here is from Pascal's triangle; in fact the three terms are multiplied by 1, 2, 1 respectively.

Pascal's Triangle

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
- - - - - - - - - - - - - - -
- - - -

4.1.3 For the third derivative we differentiate three times:

$$\begin{array}{llll} x^5 & 5x^4 & 20x^3 & 60x^2 \\ e^{2x} & 2e^{2x} & 4e^{2x} & 8e^{2x} \end{array}$$

The four products are multiplied by 1, 3, 3, 1 respectively:

$D_3 = x^5 \times 8e^{2x} + 3 \times 5x^4 \times 4e^{2x} + 3 \times 20x^3 \times 2e^{2x} + 60x^2 \times e^{2x}$.

And so on for higher derivatives.

4.1.4 This process is equivalent to Leibniz' theorem, which says that the nth derivative of a product uv is given by:

$$(uv)_n = u_0 v_n + \binom{n}{1} u_1 v_{n\text{-}1} + \binom{n}{2} u_2 v_{n\text{-}2} + \ldots$$

This gives us the required derivative, of course, but does not bring out the underlying vertical and crosswise pattern, which makes the Vedic method much more direct and practical.

4.2 Maclaurin and Taylor Theorems

4.2.1 Tirthaji opens chapter 22 as follows:

"In this Chapter the relevant Sūtras (*Guṇaka-Samuccaya* etc.,) dealing with successive differentiations, covering Leibniz theorem, Maclaurin's theorem, Taylor's theorem etc., are given and a lot of other material which is yet to be studied and decided on by the great mathematicians of the present-day western world, is also given."

As shown above we do use successive differentiations in Leibniz theorem. The way in which successive differentials are used in Maclaurin's and Taylor's theorem is explained below.

4.2.2 Maclaurin's theorem is expressed as $f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2} + f'''(0)\frac{x^3}{6} \ldots$

Here as we go from term to term we see successive derivatives of $f(0)$ and successive integrals of 1, both with respect to x. This suggests an easy way to apply the theorem.

So to use Maclaurin's Theorem to obtain a series expansion for e^{2x} we proceed as follows.

$$\begin{aligned} e^{2x} &= e^{2.0} + 2e^{2.0}x + 4e^{2.0}\tfrac{x^2}{2} + 8e^{2.0}\tfrac{x^3}{6} + \ldots \\ &= 1 + 2x + 2x^2 + \tfrac{4}{3}x^3 + \ldots \end{aligned}$$

We put down $e^{2.0}$ initially for $f(0)$ and differentiate it as if the zero were an x.
This is easy to do in practice and means we can put the series straight down, term by term, each term being obtained from the previous one.

4.2.3 We may note that since Maclaurin's Theorem can be used to derive the Binomial Theorem it is not surprising that the same pattern is present in both:
$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}b^3 \ldots$

And in fact the method extends neatly to the trinomial and multinomial theorems[3].

Taylor's theorem is a more general case of Maclaurin's Theorem which expands a function around some other point rather than zero. It is given by:

$$f(x) = f(a) + f'(a)(x-a) + f''(a)\frac{(x-a)^2}{2!} + f'''(a)\frac{(x-a)^3}{3!} + \ldots$$

Again we see successive differentials and integrations occurring together.

5. CONCLUDING REMARKS

Although the two methods given by Tirthaji in this chapter have plenty of examples it is a bit short of explanation of the logic and applications. It seems Tirthaji expects the reader to do some work themselves to understand these methods and their implications.

References

[1] Bharati Krishna Tirthaji Maharaja, (1965). Vedic Mathematics. Delhi: Motilal Banarasidas,.

[2] Williams. K. R. (2001). Discover Vedic Mathematics. U.K.: Inspiration Books (Chapter 16).

[3] Williams. K. R. (2013). The Crowning Gem. U.K.: Inspiration Books (Appendix 2).