AN INVESTIGATION INTO THE WORKING OF THE EKADHIKENA PURVENA SUTRA, AND HOW IT CAN BE USED TO IDENTIFY PRIME NUMBERS

Marianne Fletcher
The British Academy, Pretoria, South Africa

Introduction

The identification of very large prime numbers is of considerable importance in fields such as cryptology and coding in security systems. Both the ***Ekadhikena Purvena*** and ***Nikhilam Navatascaramam Dasatah*** sutras, expounded by Sri Bharati Krishna Tirtha in his book "Vedic Mathematics" can be used to quickly and efficiently generate all the digits before recurrence in the decimal string of any non-terminating rational number. Because the cyclic length of a decimal string can be used to determine whether the denominator of a particular rational number is ***prime*** or not, these Vedic Mathematics sutras can be employed as tools in the identification process of a prime number.

A simple computer program was written, which employs the computational steps outlined by the above-mentioned two sutras. The recurring decimal strings related to thousands of different rational numbers were thereby obtained and subsequently analyzed.

It was found that, if x is the number of digits before recurrence in the cyclic decimal string of a rational number $\frac{1}{N}$, and if $(N-1)$ is not divisible by x, then *N* is not a prime number. If x equals $(N-1)$, then *N* is always prime, while if $\frac{N-1}{x}$ yields a whole number greater than 1, *N* is almost always prime.

A subsequent study of basic number theory revealed that this test for primeness is known as Fermat's Primality Test. It is an application of Fermat's Little Theorem, which was stated by Pierre de Fermat as early as 1620, and was proven both by Gottfried Leibnitz in the 1680's as well as by Euler in 1736.

Fermat's Primality Test cannot be used as a fool-proof method to identify prime numbers due to the fact that a small percentage of non-primes also display divisibility of $(N-1)$ by x. This article investigates why this occurs, as well as an additional simple test which can be employed to root out these Fermat pseudo-primes, thus enabling the Fermat Primality Test to indeed be employed as a useful prime number sieve. Theorem 88 (which employs the concept of a relative prime or a co-prime) from the book "An Introduction to the Theory of Numbers" by Hardy and Wright serves as a basis to explain this further test.

The Ekadhikena Purvena sutra (in conjunction with the Nikhilam sutra) proved to be a very useful tool to help generate very long decimal strings. The analysis of these strings was of considerable help in gaining conceptual understanding of Fermat's Little Theorem and the reason for the existence of pseudo-primes and how they can successfully be "sieved" when applying Fermat's Primality Test. So, although possibly nothing new (in terms of existing number theory) was found in this empirical investigation, the Ekadhikena Purvena Vedic mathematics sutra is clearly shown to be a powerful algorithm and investigative tool.

An additional finding involves a concept often employed in Vedic Mathematics, namely that of a ***digital root***. It appears that, in the case of a recurring decimal string containing an ***odd number*** of digits in the cyclic string, the ***value of the digital root*** of all the digits in the string can also be used to help sift primes from non-primes.

Some important terms and concepts

1) A ***prime number*** is a number only divisible by 1 and itself, e.g. 2, 3, 5, 7, 11, 13, 17,19, 23, 29 etc.

2) A ***rational number*** is any number which can be written in the form a/b (where $b \neq 0$). Rational numbers can be divided into three categories:

   Category 1: A rational number has a ***terminating*** decimal string if its denominator is 2 or 5 or a product of powers of 2's and 5's e.g.

   ½ = 0.5   1/16 = $½^4$ = 0.0625   1/1000 = 1/(2^3x5^3) = 0.00   11/5 = 0.2

   1/125 = 1/5^3 = 0.008   3/2000 = 1/(2^4x5^3) = 0.0065   7/500 = 7/(2^2x5^3) = 0.014

   If the decimal string of a rational number is ***non-terminating***, it displays either ***perfect recurrence*** or non-perfect recurrence:

   Category 2: ***Non-perfect recurrence*** refers to there being both a non-recurring and recurring part to the string. Such strings are generated when the denominator of the rational number is a product of a prime (or primes) other than 2 or 5, with powers of either 2 or 5 or both, e.g.

   1/6 = ½ x 1/3 = 0.1666…   1/75 = 1/(5^2 x 3) = 0.01333…

   1/35 = 1/5 x 1/7 = 0.0285714 285714 …   (The single 0 after the decimal point is non-recurring.)

   Category 3: However, a ***perfectly recurring*** decimal string is generated when the denominator is either a *prime number* (other than 2 or 5), or a product of powers of prime numbers (other than 2 or 5).

   Some examples:

   1/3 = 0.333…   1/7 = 0.142857 142857…   1/11 = 0.090909…

   1/13 = 0.076923 076823…   1/19 = 0.052631578947368421 052631578947368421…

   It follows that any prime number (other than 2 or 5) always ends on one of the digits 1, 3, 7 or 9.

3) ***The Ekadhikena Purvena sutra only applies to rational numbers in Category 3.*** Hence, only perfectly recurring decimal strings related to rational numbers with denominators ending on 1, 3, 7 or 9 can be generated using this sutra. This covers all possible prime numbers excluding 2 and 5.

4) The ***digital root (DR)*** of a number is obtained by finding the sum of all the digits in the number. If this answer consists of more than one digit, all these digits are again summed. This process is repeated until only a single digit remains.

   For example: For 2439:   2 + 4 + 3 + 9 = 18
   Then for 18:   1 + 8 = 9
   Thus the digital root (DR) of 2439 is 9

The Ekadhikena Purvena and Nikilam Sutras applied to 1/19

Employing a conventional long division method, the recurring decimal string for, say, 1/19 is found as follows:

```
         0. 0 5 2 6 3 1 5 7 8 9 4 7 3 6 8 4 2 1   0 5 2 6…..
      19/ 1. 0 0 0 0
          0  0
          1  0 0
             9 5
               5 0
               3 8
               1 2 0
               1 1 4
                   60 … etc.
```

Or in one line:

$$\begin{array}{l} \quad\; \underline{0.\,0\;\;5\;\;2\;\;6\;\;3\,1\;\;\;5\;\;\;7\;\;8\;\;\;9\,4\;\;\;7\;\;3\;\;6\;\;8\,4\,2\,1\;\;\;\;0\;\;5\;\;2\;\;6\ldots} \\ 19\;/\;1.{}^{1}0^{10}0^{5}0^{12}0^{6}0^{3}0^{11}0^{15}0^{17}0^{18}0^{9}0^{14}0^{7}0^{13}0^{16}0^{8}0^{4}0^{2}0\;\;{}^{1}0^{10}0^{5}0^{12}0\ldots \end{array}$$

The process proceeds by the following steps:

$$\frac{1}{19} = 0 + \left(\frac{1}{19}\right)$$
$$\frac{1}{19} = 0 + \left(\frac{10}{19}\right)(0{,}1)^1$$
$$\frac{1}{19} = 0 + \left(0 + \frac{10}{19}\right)(0{,}1)^1$$
$$\frac{1}{19} = 0 + \ 0(0{,}1)^1 + \left(\frac{100}{19}\right)(0{,}1)^2$$
$$\frac{1}{19} = 0 + \ 0(0{,}1)^1 + \left(5 + \frac{5}{19}\right)(0{,}1)^2$$
$$\frac{1}{19} = 0 + \ 0(0{,}1)^1 + 5(0{,}1)^2 + \left(\frac{50}{19}\right)(0{,}1)^3$$
$$\frac{1}{19} = 0 + \ 0(0{,}1)^1 + 5(0{,}1)^2 + \left(\frac{50}{19}\right)(0{,}1)^3$$
$$\frac{1}{19} = 0 + \ 0(0{,}1)^1 + 5(0{,}1)^2 + \left(2 + \frac{12}{19}\right)(0{,}1)^3$$
$$\frac{1}{19} = 0 + \ 0(0{,}1)^1 + 5(0{,}1)^2 + 2(0{,}1)^3 + \left(\frac{120}{19}\right)(0{,}1)^4$$
$$\frac{1}{19} = 0 + \ 0(0{,}1)^1 + 5(0{,}1)^2 + 2(0{,}1)^3 + \left(6 + \frac{6}{19}\right)(0{,}1)^4$$
$$\frac{1}{19} = 0 + \ 0(0{,}1)^1 + 5(0{,}1)^2 + 2(0{,}1)^3 + 6(0{,}1)^4 + \left(\frac{60}{19}\right)(0{,}1)^5$$
$$\frac{1}{19} = 0 + \ 0(0{,}1)^1 + 5(0{,}1)^2 + 2(0{,}1)^3 + 6(0{,}1)^4 + \left(3 + \frac{3}{19}\right)(0{,}1)^5$$
$$\frac{1}{19} = 0 + \ 0(0{,}1)^1 + 5(0{,}1)^2 + 2(0{,}1)^3 + 6(0{,}1)^4 + 3(0{,}1)^5 + \left(\frac{30}{19}\right)(0{,}1)^6$$
$$\frac{1}{19} = 0 + \ 0(0{,}1)^1 + 5(0{,}1)^2 + 2(0{,}1)^3 + 6(0{,}1)^4 + 3(0{,}1)^5 + \left(1 + \frac{11}{19}\right)(0{,}1)^6$$
$$\frac{1}{19} = 0 + \ 0(0{,}1)^1 + 5(0{,}1)^2 + 2(0{,}1)^3 + 6(0{,}1)^4 + 3(0{,}1)^5 + 1(0{,}1)^6 + \left(\frac{110}{19}\right)(0{,}1)^7 \text{ etc.}$$

Thus $\frac{1}{19} = 0{,}052631\ldots$ etc.

This process is repeated ***until a remainder of 1 is once again reached***, i.e.

$$\frac{1}{19} = 0 + 0(0{,}1)^1 + 5(0{,}1)^2 + 2(0{,}1)^3 + 6(0{,}1)^4 + \ \ldots\ldots + 4(0{,}1)^{16} + 2(0{,}1)^{17} + 1(0{,}1)^{18} + \left(\frac{1}{19}\right)(0{,}1)^{19}$$

Because “1” is the very first dividend, any further steps in the division process yield exactly the same sequence of digits again, i.e.

0 5 2 6 3 1 5 7 8 9 4 7 3 6 8 4 2 1

There are x =18 digits before recurrence.

(Note: When doing conventional long division to find the perfectly recurring decimal string for $\frac{1}{N}$, recurrence is always reached when a remainder of 1 is obtained.)

Sri Tirtha showed that the recurring decimal string can be found far more easily (compared to the above conventional method) by using the ***Ekadhikena Purvena*** division technique.

The Ekadhikena Purvena sutra simply states:

"By one more than the previous one".

For 1/19, the denominator consists of the two digits 1 and 9. Defining the "previous one" as the digit before the "9", i.e. the "1" in the case of 19: "One more than the previous one" is:

$$1 + 1 = 2$$

This 2 (called the "ekadhika") is now the new divisor (from left to right), or also the new multiplier (from right to left).

For string generation ***from left to right***, instead of attempting to divide 19 into 1, (according to the conventional method), the procedure is now to simply divide ***2*** into 1 instead, i.e.

2 divided into 1 equals 0 remainder 1. For this, write:

$${}^{1}0$$

with the rem 1 a superscript to the left of the quotient 0. Then divide 2 into 10, giving 5 rem 0:

$${}^{1}0\ {}^{0}5$$

Then divide 2 into 5, giving 2 rem 1:

$${}^{1}0\ {}^{0}5\ {}^{1}2$$

Division of 2 into 12 then yields 6 rem 0:

$${}^{1}0\ {}^{0}5\ {}^{1}2\ {}^{0}6$$

Division of 2 into 6 yields 3 rem 0:

$${}^{1}0\ {}^{0}5\ {}^{1}2\ {}^{0}6\ {}^{0}3$$

Division of 2 into 3 yields 1 rem 1:

$${}^{1}0\ {}^{0}5\ {}^{1}2\ {}^{0}6\ {}^{0}3\ {}^{1}1$$

Division of 2 into 11 yields 5 rem 1:

$${}^{1}0\ {}^{0}5\ {}^{1}2\ {}^{0}6\ {}^{0}3\ {}^{1}1\ {}^{1}5$$

Proceeding thus, the 18 digits of the recurring string

$${}^{1}0\ {}^{0}5\ {}^{1}2\ {}^{0}6\ {}^{0}3\ {}^{1}1\ {}^{1}5\ {}^{1}7\ {}^{1}8\ {}^{0}9\ {}^{1}4\ 7\ {}^{1}3\ {}^{1}6\ {}^{0}8\ {}^{0}4\ {}^{0}2\ {}^{0}1$$ are generated.

Alternatively, the digits in the decimal string can also be generated ***from right to left*** in the following way:

Starting with 1 ***on the very right***, multiply the 1 by 2 to obtain 2; then multiply this product by 2 again to obtain 4; then multiply 4 by 2 to obtain 8; then multiply 8 by 2 to get 16, i.e.

$${}^{1}6\ 8\ 4\ 2\ 1$$

where the ten's digit of the 16 is written as a superscript 1, ready to be added ("carried over") onto the product of the next multiplication by 2. The next step is to multiply only the 6 by 2 to get 12, after which the superscript 1 (from the ten's digit of 16) is added onto 12 to get 13, i.e.

$${}^{1}3\ 6\ 8\ 4\ 2\ 1$$

Now multiply only the 3 by 2, then add 1 to get 7. Multiply 7 by 2 to get 14, and write the ten's digit of the 14 as a superscript 1:

$${}^{1}4\ 7\ {}^{1}3\ {}^{1}6\ 8\ 4\ 2\ 1$$

Proceeding thus, the complete cyclic string is obtained:

$${}^{1}0\ 5\ {}^{1}2\ 6\ 3\ {}^{1}1\ {}^{1}5\ {}^{1}7\ {}^{1}8\ 9\ {}^{1}4\ 7\ {}^{1}3\ {}^{1}6\ 8\ 4\ 2\ 1$$

In the case of 1/19, the number of steps in the calculation can, furthermore, be ***halved*** by noting that the string of digits comprising the first half of the decimal expansion, added to the string of digits making up the second half, yields a ***sequence of nines***, i.e.

$$\begin{array}{r} 052631578 \\ \underline{947368421} \\ \underline{999999999} \end{array}$$

This phenomenon is an application of the ***Nikhilam Navatascaramam Dasatah*** sutra:

"All from 9 and the last from 10"

because when the digits in the first half of the string are subtracted from 9, the digits in the second half of the string are obtained. "The last from 10" never features, as there is no last digit in a non-terminating string.

Explanation of the working of the Ekadhikena Purvena Sutra:

The recurring decimal string associated with a fraction is but a ***geometric sequence*** of the numbers generated by dividing or multiplying successive terms by a common ratio related to the "ekadhika".

To demonstrate this, all the steps in the ***conventional*** division process are set out below:

Step 0:	1/19	=	0 + 1/19
Step 1:	10/19	=	0 + 10/19
Step 2:	100/19	=	5 + 5/19
Step 3:	50/19	=	2 + 12/19
Step 4:	120/19	=	6 + 6/19
Step 5:	60/19	=	3 + 3/19
Step 6:	30/19	=	1 + 11/19
Step 7:	110/19	=	5 + 15/19
Step 8:	150/19	=	7 + 17/19
Step 9:	170/19	=	8 + 18/19
Step 10:	180/19	=	9 + 9/19
Step 11:	90/19	=	4 + 14/19
Step 12:	140/19	=	7 + 7/19
Step 13:	70/19	=	3 + 13/19
Step 14:	130/19	=	6 + 16/19
Step 15:	160/19	=	8 + 8/19
Step 16:	80/19	=	4 + 4/19
Step 17:	40/19	=	2 + 2/19
Step 18:	20/19	=	1 + 1/19

Each equation above can be multiplied by the divisor 19, to yield equations in the form:

Dividend = Quotient x (Divisor) + Remainder or Dividend = Quotient x (19) + Rem

Step 0:	1	=	0(19) + 1					
Step 1:	10	=	0(19) + 10			= 0(19 + 1) + 10		= 0(20) + 10
Step 2:	100	=	5(19) + 5			= 5(19 + 1)		= 5(20)
Step 3:	50	=	2(19) + 12	=	2(19) + 2 + 10	= 2(19 + 1) + 10		= 2(20) + 10
Step 4:	120	=	6(19) + 6			= 6(19 + 1)		= 6(20)
Step 5:	60	=	3(19) + 3			= 3(19 + 1)		= 3(20)
Step 6:	30	=	1(19) + 11	=	1(19) + 1 + 10	= 1(19 + 1) + 10		= 1(20) + 10
Step 7:	110	=	5(19) + 15	=	5(19) + 5 + 10	= 5(19 + 1) + 10		= 5(20) + 10
Step 8:	150	=	7(19) + 17			= 7(19 + 1)		= 7(20)
Step 9:	170	=	8(19) + 18	=	8(19) + 8 + 10	= 8(19 + 1) + 10		= 8(20) + 10
Step 10:	180	=	9(19) + 9			= 9(19 + 1)		= 9(20)
Step 11:	90	=	4(19) + 14	=	4(19) + 4 + 10	= 4(19 + 1) + 10		= 4(20) + 10
Step 12:	140	=	7(19) + 7			= 7(19 + 1)		= 7(20)
Step 13:	70	=	3(19) + 13	=	3(19) + 3 + 10	= 3(19 + 1) + 10		= 3(20) + 10
Step 14:	130	=	6(19) + 16	=	6(19) + 6 + 10	= 6(19 + 1) + 10		= 6(20) + 10
Step 15:	160	=	8(19) + 8			= 8(19 + 1)		= 8(20)
Step 16:	80	=	4(19) + 4			= 4(19 + 1)		= 4(20)
Step 17:	40	=	2(19) + 2			= 2(19 + 1)		= 2(20)
Step 18:	20	=	1(19) + 1			= 1(19 + 1)		= 1(20)

As demonstrated above, each equation can then also be rewritten in the form:

Dividend = Quotient x (N + 1) + Rem

thus Dividend = Quotient x (20) + Rem

where 20/10 is the "ekadhika" 2.

Studying the right hand column from ***bottom to top***, the working of the ***right to left multiplication*** process of the Ekadhikena sutra is revealed i.e. 1(2) = 2; then 2(2) = 4, then 4(2) = 8, etc. The string is therefore ***a geometric sequence*** generated from ***right to left*** by multiplying a common ratio 2 with successive terms, starting at a term equal to 1.

Should each of the above equations now be ***divided*** by 20, then a perusal of the right hand column (see below) from ***top to bottom***, shows the working of the ***left to right division*** process of the Ekadhikena Purvena sutra, i.e. 10/2 = 5; 5/2 = 2 rem 1; 12/2 = 6 rem 0, etc.

$$\frac{10}{20} = \frac{1}{2} = \frac{0}{20} + \frac{10}{20} = 0 + \frac{1}{2} \equiv \text{0 rem 1}$$

$$\frac{100}{20} = \frac{10}{2} = 5 + \frac{0}{20} = 5 \equiv \text{5 rem 0}$$

$$\frac{50}{20} = \frac{5}{2} = 2 + \frac{10}{20} = 2 + \frac{1}{2} \equiv \text{2 rem 1}$$

$$\frac{120}{20} = \frac{12}{2} = 6 + \frac{0}{20} = 6 \equiv \text{6 rem 0}$$

$$\frac{60}{20} = \frac{6}{2} = 3 + \frac{0}{20} = 3 \equiv \text{3 rem 0}$$

$$\frac{30}{20} = \frac{3}{2} = 1 + \frac{10}{20} = 1 + \frac{1}{2} \equiv \text{1 rem 0}$$

$$\frac{110}{20} = \frac{11}{2} = 5 + \frac{10}{20} = 5 + \frac{1}{2} \equiv \text{5 rem 1}$$

$$\frac{150}{20} = \frac{15}{2} = 7 + \frac{10}{20} = 7 + \frac{1}{2} \equiv \text{7 rem 1 etc.}$$

Thus it is again demonstrated that the string is a geometric sequence, generated from ***left to right*** by multiplying a common ratio of $\frac{1}{2}$ with successive terms, starting at a term equal to 1.

The "ekadhika" (which is 2 in the case for 1/19) can also be identified by applying the formula for the sum to infinity S_n of a geometric sequence to the particular fraction, i.e.

$$S_n = \frac{a}{1-r} = \frac{1}{19} = \frac{1}{20-1} = \frac{\frac{1}{20}}{1-\frac{1}{20}}$$

where *a* is the first term, and *r* is the common ratio.

For 1/19, $a = \frac{1}{20}$ and $r = \frac{1}{20}$.

Because $S_n = a + ar + ar^2 + ar^3 + ar^4 + ar^5 + \ldots\ldots\ldots$

it follows that $\frac{1}{19} = \left(\frac{1}{20}\right) + \left(\frac{1}{20}\right)^2 + \left(\frac{1}{20}\right)^3 + \left(\frac{1}{20}\right)^4 + \left(\frac{1}{20}\right)^5 + \ldots$

thus $\frac{1}{19} = \sum_{n=1}^{\infty} \left(\frac{1}{20}\right)^n$

or $\frac{1}{19}$ = 0.05 +
0.0025 +
0.000125 +
0.00000625 +
0.0000003125 +
0.000000015625 +
0.00000000078125 +
<u>0.0000000000390625</u> + … etc.
<u>0.0526315789….</u>

In general, ***any*** perfectly recurring decimal for 1/N can be written as: $\frac{1}{N} = \sum_{n=1}^{\infty}\left(\frac{1}{N+1}\right)^{n}$
The ekadhika is then (N+1)/10.

The Ekadhikena Purvena Sutra applied to 1/N where N has a final digit 1, 3 or 7:
When the sutra refers to the "previous one" it must always be "previous to" the digit "9" in the denominator of a fraction.

So that the sutra can be applied to rational numbers (in form a/b) with denominators ending also on the digits 1, 3 and 7, such fractions can be manipulated as follows to have a last digit equal to 9:

$$\frac{1}{21} = \frac{1}{21}\times\frac{9}{9} = \frac{9}{189}\frac{1}{13} = \frac{1}{13}\times\frac{3}{3} = \frac{3}{39} \qquad \frac{1}{7} = \frac{1}{7}\times\frac{7}{7} = \frac{7}{49}$$

The Ekadhikena Purvena Sutra Applied to 1/13

$$\frac{1}{13} = \frac{1}{13}\times\frac{3}{3} = \frac{3}{39}$$

For 39, "one more than the previous one" is 3 + 1 = 4. The ekadhika is thus 4. For ***right to left*** string generation, start with the numerator 3 as the last digit before recurrence, and then multiply successively with 4, thereby obtaining:

$^{3}0\ ^{2}7\ ^{3}6\ 9\ ^{1}2\ 3$

Thus $\boxed{1/13 = 0.076923\ 076923 \ldots \text{rec}}$

This process is thus repeated until a remainder of 3 is once again reached. Because "3" is the very first multiplicand, any further steps in the process yield exactly the same sequence of digits again.

Also: 076 +
<u>923</u>
<u>999</u>

Note: Employing $\frac{1}{N} = \sum_{n=1}^{\infty}\left(\frac{1}{N+1}\right)^{n}$

3/39 = 3 x $\frac{1}{39} = 3\times\sum_{n=1}^{\infty}\left(\frac{1}{40}\right)^{n}$

The ekadhika = 40/10 = 4

The Ekadhikena Purvena Sutra Applied to 1/7

$$\frac{1}{7} = \frac{1}{7} \times \frac{7}{7} = \frac{7}{49}$$

For 49, "one more than the one before" is 4 + 1 = 5. The ekadhika is thus 5. For ***right to left*** string generation, start with the numerator 7 as the last digit before recurrence, and then multiply successively with 5, thereby obtaining:

$^{2}1\ ^{1}4\ ^{4}2\ ^{2}8\ ^{3}5\ 7$

Thus $\boxed{1/7 = 0.142857\ 142857 \ldots \text{ rec}}$

This process is thus repeated until a remainder of 7 is once again reached. Because "7" is the very first multiplicand, any further steps in the process yield exactly the same sequence of digits again.

Employing $\frac{1}{N} = \sum_{n=1}^{\infty} \left(\frac{1}{N+1}\right)^{n}$

$7/49 = 7 \times \frac{1}{49} = 7 \times \sum_{n=1}^{\infty} \left(\frac{1}{50}\right)^{n}$

The ekadhika = 50/10 = 5

Also: $\begin{array}{r} 142\ + \\ \underline{857} \\ \underline{\underline{999}} \end{array}$

The Ekadhikena Purvena Sutra Applied to 1/21

$$\frac{1}{21} = \frac{1}{21} \times \frac{9}{9} = \frac{9}{189}$$

For 189, "one more than the one before" is 18 + 1 = 19. The ekadhika is thus 19. For ***right to left*** string generation, start with the numerator 9 as the last digit before recurrence, and then multiply successively with 19, thereby obtaining:

$^{9}0\ ^{14}4\ ^{11}7\ ^{3}6\ ^{17}1\ \ 9$

Thus $\boxed{1/21 = 0.047619\ 047619 \ldots \text{ rec}}$

This process is thus repeated until a remainder of 9 is once again reached. Because "9" is the very first multiplicand, any further steps in the process yield exactly the same sequence of digits again.

Employing $\frac{1}{N} = \sum_{n=1}^{\infty} \left(\frac{1}{N+1}\right)^{n}$

$9/189 = 9 \times \frac{1}{189} = \sum_{n=1}^{\infty} \left(\frac{1}{190}\right)^{n}$

The ekhaddhika = 190/10 = 19

Here: $\begin{array}{r} 047\ + \\ \underline{619} \\ \underline{\underline{666}} \end{array}$

Note: For an *N* value ending on the digit 9, the last digit in the recurring string for $\frac{1}{N}$ is "1". However for *N* values ending, respectively, on the digits 7, 3 and 1, the last digits in the respective recurring decimal strings for $\frac{1}{N}$ are 7, 3 and 9. This is, of course, due to the fact that the $\frac{1}{N}$ values have been multiplied by 7, 3 or 9 in order to apply the Ekadhikena sutra.

Results of applying the sutras to $\frac{1}{N}$ values for N = 3 to N = 10000

A simple computer program was written which employs the computational steps outlined by the Ekadhikena Purvena as well as the Nikilam sutra. The recurring decimal string for $\frac{1}{N}$ was subsequently calculated for all *N* values (ending on the digits 1, 3, 7 or 9) between 3 and 10000.

The calculation process involved a continuous scanning to check whether - upon addition of their complements from 9 - successive digits in the string started to yield a string of 9's. When this tendency was confirmed, the Ekadhikena process was terminated, as the rest of the string could be found more easily by application of the Nikilam sutra.

For each decimal string, the following was determined:

1) All the digits in the full decimal string before recurrence;
2) The number of digits x in the cyclic string;
3) A value $k = \frac{N-1}{x}$;
4) Each value *N* was compared against a known list of primes to ascertain its prime status.

Some results (only for *N* values up to 389) are displayed in Table 1 and Table 2.

Some *DR* (digital root) values for ***odd number*** x values (when all the digits in one cycle of a recurring decimal string are added together) are also listed in Table 1.

Table 1: Analysis of Recurring Decimal Strings For Some Selected *N*-values

N	$\frac{1}{N}$	First half added to second half	*DR*	p or n	*N* - 1	x	$\frac{N-1}{x}$
7	$\frac{1}{7} = 0.\dot{1}4285\dot{7}$	142 857 999		p	6	6	1
13	$\frac{1}{13} = 0.\dot{0}7692\dot{3}$	076 923 999		p	12	6	2
17	$\frac{1}{17} = 0.\dot{0}58823529411764\dot{7}$	05882352 94117647 99999999		p	16	16	1
19	$\frac{1}{19} = 0.\dot{0}5263157894736842\dot{1}$	052631578 947368421 999999999		p	18	18	1
21	$\frac{1}{21} = 0.\dot{0}4761\dot{9}$	047 619 666		n	20	6	$3.\dot{3}$
23	$\frac{1}{23} = 0.\dot{0}43478260869565217391\dot{3}$	04347826086 95652173913 99999999999		p	22	22	1
27	$\frac{1}{27} = 0.\dot{0}3\dot{7}$	x not even	1	n	26	3	$8.\dot{6}$
37	$\frac{1}{37} = 0.\dot{0}2\dot{7}$	x not even	9	p	6	3	12
41	$\frac{1}{41} = 0.\dot{0}243\dot{9}$	x not even	9	p	40	5	8
43	$\frac{1}{43} = 0.\dot{0}2325581395348837209\dot{3}$	x not even	9	p	42	21	2
51	$\frac{1}{51} = 0.\dot{0}19607843137254\dot{9}$	01960784 31372549 33333333		n	50	16	3.125
69	$\frac{1}{69} = 0.\dot{0}14492753623188405797\dot{1}$	01449275362 31884057971 33333333333		n	68	22	$30.\dot{0}\dot{9}$
79	$\frac{1}{79} = 0.\dot{0}12658227848\dot{1}$	x not even	9	p	78	13	6
81	$\frac{1}{81} = 0.\dot{0}1234567\dot{9}$	x not even	1	n	80	9	$8.\dot{8}$
91	$\frac{1}{91} = 0.\dot{0}1098\dot{9}$	010 989 999		n	90	6	15
133	$\frac{1}{133} = 0.\dot{0}0751879699248120\dot{3}$	007518796 992481203 999999999		n	132	18	$7.\dot{3}$

Table 2: The Number of Digits x in Decimal Strings for *N*-values between 3 and 389

N		x	k	N		x	k	N		x	k
				131	p	130	1	261	n	28	9.285..
3	p	1	2	133	n	18	7.333..	263	p	262	1
7	p	6	1	137	p	8	17	267	n	44	6.045..
9**	n	1	8	139	p	46	3	269	p	268	1
11	p	2	5	141	n	46	3.043..	271	p	5	54
13	p	6	2	143	n	6	23.66..	273	n	6	45.33..
17	p	16	1	147	n	42	3.476..	277	p	69	4
19	p	18	1	149	p	148	1	279	n	15	18.53..
21	n	6	3.33..	151	p	75	2	281	p	28	10
23	p	22	1	153	n	16	9.5	283	p	141	2
27	n	3	8.66..	157	p	78	2	287	n	30	9.533..
29	p	28	1	159	n	13	12.15..	289	n	272	1.058..
31	p	15	2	161	n	66	2.42..	291	n	96	3.020..
33	n	2	16	163	p	81	2	293	p	146	2
37	p	3	12	167	p	166	1	297	n	6	49.33..
39	n	6	6.33..	169	n	78	2.15..	299	n	66	4.515..
41	p	5	8	171	n	18	9.44..	301	n	42	7.142..
43	p	21	2	173	p	43	4	303	n	4	75.5
47	p	46	1	177	n	58	3.034..	307	p	153	2
49	n	42	1.14..	179	p	178	1	309	n	34	9.058..
51	n	16	3.125	181	p	180	1	311	p	155	2
53	p	13	4	183	n	60	3.033..	313	p	312	1
57	n	18	3.111..	187	n	16	11.625	317	p	79	4
59	p	58	1	189	n	6	31.33..	319	n	28	11.35..
61	p	60	1	191	p	95	2	321	n	53	6.037..
63	n	6	10.33..	193	p	192	1	323	n	144	2.236..
67	p	33	2	197	p	98	2	327	n	108	3.018..
69	n	22	3.090..	199	p	99	2	329	n	138	2.376..
71	p	35	2	201	n	33	6.060..	331	p	110	3
73	p	8	9	203	n	84	2.404..	333	n	3	110.6..
77	n	6	12.66..	207	n	22	9.363..	337	p	336	1
79	p	13	6	209	n	18	11.55..	339	n	112	3.017..
81	n	9	8.88..	211	p	30	7	341	n	30	11.33..
83	p	41	2	213	n	35	6.057..	343	n	294	1.163..
87	n	28	3.071..	217	n	30	7.2	347	p	173	2
89	p	44	2	219	n	8	27.25	349	p	116	3
91**	n	6	15	221	n	48	4.583..	351	n	6	58.33..
93	n	15	6.133..	223	p	222	1	353	p	32	11
97	p	96	1	227	p	113	2	357	n	48	7.416..
99**	n	2	49	229	p	228	1	359	p	179	2
101	p	4	25	231	n	6	38.33..	361	n	342	1.052..
103	p	34	3	233	p	232	1	363	n	22	16.45..
107	p	53	2	237	n	13	18.15..	367	p	366	1
109	p	108	1	239	p	7	34	369	n	5	73.6
111	n	3	36.66..	241	p	30	8	371	n	78	4.743..
113	p	112	1	243	n	27	8.96..	373	p	186	2
117	n	6	19.33..	247	n	18	13.66..	377	n	84	4.476..
119	n	48	2.458..	249	n	41	6.048..	379	p	378	1
121	n	22	5.454..	251	p	50	5	381	n	42	9.047..
123	n	5	24.4	253	n	22	11.45..	383	p	382	1
127	p	42	3	257	p	256	1	387	n	21	18.38..
129	n	21	6.095..	259**	n	6	43	389	p	388	1

* $\frac{1}{119} = 0.\dot{0}08403361344537815126050420168067226890756302521\dot{1}$

008403361344537815126050 +
420168067226890756302521
428571428571428571428571

(1/119 is a non-prime with x = 48 digits in a cyclic string.)

Observations from the results

1) Whenever $N - 1 = x$, i.e. ($N - 1$) is exactly divisible by its number of recurring decimals, yielding $k = \frac{N-1}{x} = 1$, then N is ***always prime***.

2) Whenever $k = \frac{N-1}{x}$ has a whole number value greater than 1, i.e. whenever ($N - 1$) is found to be a multiple (other than 1) of its number of recurring decimals, then N is almost always prime, but there are some exceptions, called "pseudo-primes". Within the range investigated, the exceptions only occur for $k \geq 8$.

3) ***When the number of digits x in one cycle of the recurring decimal string is an even number,*** the process of adding the digits in the first half of the string to the digits in the second half, always yields a new recurring pattern:

   (a) In the case of ***primes***, this recurring pattern ***always consists of a string of 9's*** (e.g. for 1/7 and 1/19).
   (b) However, in the case of ***non-primes***, sometimes the two half strings indeed add to yield a string of 9's (e.g. for 1/91 – a pseudo-prime!); but often they add to yield a string of recurring 3's (e.g. for 1/51 and 1/69) or recurring 6's (e.g. for 1/21) or even a string made up of the pattern 142857 repeating over and over again (e.g. for 1/119 – see below*). See Note 2 in the addendum for a further discussion of this phenomenon.

4) ***When the number of digits x in one cycle of the recurring string is an odd number***, the string cannot be divided exactly into halves, thus no summation of equal halves can be done. However, in such cases calculation of the ***digital root*** of the full string yields the following results within the range investigated:

   (a) For a ***prime*** the digital root always equals 9.
   (b) However, for a ***non-prime*** the digital root can be 1, 3, 5, 6, 7 or 9.

5) Within the range of numbers investigated, the longest string of digits which needed to be generated before recurrence was found, was in the case of N = 9967 (a prime) which has 9966 (= $N - 1$) digits in the string for 1/N. However, some numbers, such as N = 7471 (a non-prime with only 30 digits in its 1/N string) required far less computational steps.

6) Analysis of the number of times k = 1, k = 2, k = 3, etc. occurs for ***primes*** between 3 and 10000 yields the following approximate results:

k = 1 40% of all cases	k = 3 6% occurrence	k = 5 2% occurrence
k = 2 27% of all cases	k = 4 6% occurrence	k = 6 5% occurrence

Discussion of results

All prime numbers investigated were found to have whole number k–values (where $k = \frac{N-1}{x}$) either greater or equal to 1. Thus all primes N appear to have the property that x is a divisor of N - 1. The question thus arose whether this fact could be used as a "sieve" to separate primes from non-primes. A subsequent study of number theory revealed that such a "sieve" - called Fermat's Primality Test has indeed existed for several hundred years. A brief discussion of Fermat's Little Theorem is given in the next section.

However, a small minority of non-primes (31 of them within the range investigated, i.e. below 10000) were also found to have integer k-values (though all only with $k \geq 8$). (Below 10000 there are 1229 prime numbers.)

These 31 "culprits" below 10000 have prime factorizations: ab, a^2b as well as abc.

As will soon be explained, by subjecting all N-values with integer k's greater than 1 to a further short test, these "pseudo-primes" can be eliminated with relative ease.

Table 3 ***The 31 "culprits" (or Fermat pseudo-primes base 10) below 10000***

N_c	x	$k = \frac{N-1}{x}$	Prime Factors of N_c	$a = dx + 1$	$\Phi(N)$	n_p	$\frac{n_p}{x}$
9	1	8	3,3	(2)x + 1 = 3	6	2	2
33	2	16	3,11	(1)x + 1 = 3	20	12	6
91	6	15	7,13	(1)x + 1 = 7	72	18	3
99	2	49	3,3,11	(1)x + 1 = 3	60	38	19
259	6	43	7,37	(1)x + 1 = 7	216	42	7
451	10	45	41,11	(1)x + 1 = 11	400	50	5
481	6	80	13,37	(2)x + 1 = 13	432	48	8
561	16	35	3,11,17	(1)x + 1 = 17	320	240	15
657	8	82	3,3,73	(1)x + 1 = 9	432	224	28
703	18	39	19,37	(1)x + 1 = 19	648	54	3
909	4	227	3,3,101	(2)x + 1 = 9	600	308	77
1233	8	154	3,3,137	(1)x + 1 = 9	816	416	52
1729	18	96	7,13,19	(1)x + 1 = 19	1296	432	24
2409	8	301	3,11,73	(4)x + 1 = 33	1440	968	121
2821	30	94	7,13,31	(1)x + 1 = 31	2160	660	22
2981	10	298	11, 271	(1)x + 1 = 11	2700	280	28
3333	4	833	3,11, 101	(8)x + 1 = 33	2000	1332	333
3367	6	561	7, 13, 37	(1)x + 1 = 7	2592	774	129
4141	20	207	41, 101	(2)x + 1 = 41	4000	140	7
4187	13	322	53, 79	(4)x + 1 = 53	4056	130	10
4521	8	565	3,11,137	(4)x + 1 = 33	2720	1800	225
5461	42	130	43, 127	(1)x + 1 = 43	5292	168	4
6533	46	142	47, 139	(1)x + 1 = 47	6348	184	4
6541	30	218	31, 211	(1)x + 1 = 31	6300	240	8
6601	330	20	7, 23, 41	(1/15)x + 1 = 23	5280	1320	4
7107	374	19	3, 23, 103	(1/17)x + 1 = 23	4488	2618	7
7471	30	249	31, 241	(1)x + 1 = 31	7200	270	9
7777	12	648	7, 11, 101	(1/2)x + 1 = 7	6000	1776	148
8149	28	291	29, 281	(1)x + 1 = 29	7840	308	11
8911	198	45	7, 19, 67	(1/3)x + 1 = 67	7128	1782	9

Fermat's Little Theorem (given without proof)

"If *p* is a prime number, then for any integer *a*, the number $a^p - a$ is an integer multiple of *p*."

Furthermore, if *a* is not divisible by *p*, this theorem is equivalent to the statement that $a^{p-1} - 1$ is an integer multiple of *p*. In modular arithmetic this last statement can be written as:

$$a^{p-1} \equiv 1 \pmod{p}$$

Thus, if *p* is prime and *a* is not divisible by *p*, then, $\frac{a^{p-1}}{p}$ has a remainder of 1.

This is the reason why all primes are found to have integer *k*-values: Because recurrence in $\frac{a^{p-1}}{p}$ (and thus also in $\frac{1}{p}(a^{p-1})$) occurs whenever a remainder of 1 is reached, *p* -1 must either equal the number of recurring digits x in the decimal string generated by $\frac{a^{p-1}}{p}$, or *p* -1 must be a multiple of x. Thus for a prime, *p* – 1 = $k\,x$, thus $k = \frac{p-1}{x}$ where k is an integer.

For example, for *a* = 10 and *p* = 7: then $\frac{10^6}{7} = \frac{1}{7}(10^6)$ must have a remainder 1.

This is indeed the case, as $\frac{10^6}{7}$ = 142857 + $\frac{1}{7}$ where remainder = 1.

Thus also

$$\begin{aligned} 10^6 &= 142857(7) + 1 \\ 10^6 - 1 &= 142857(7) \\ 999999 &= 142857(7) \end{aligned}$$

Thus $10^6 - 1$ is indeed a multiple of 7 as stated by the theorem.

It follows also that $\frac{1}{7}$ = 0,142857 + $\frac{1}{7}(10^{-6})$

Also for $\frac{10^{18}}{19}$ = 52631578947368421 + $\frac{1}{19}$ where remainder = 1 when x = 18 = *p* - 1

And thus also $\frac{1}{19}$ = 0,052631578947368421 + $\frac{1}{19}(10^{-18})$

The number of recurring digits can also be sub-multiples of (*p*-1), i.e. $x = \frac{p-1}{k}$ with *k* >1. Such is the case for $\frac{10^6}{13}$ = 76923 + $\frac{1}{13}$ where remainder = 1, but $x = \frac{p-1}{k} = \frac{13-1}{2} = 6$

Is the converse of Fermat's Little Theorem necessarily true, i.e. if $a^{p-1} - 1$ is an integer multiple of *p* (or stated differently: if *N* – 1 is divisible by its number of recurring digits) does this necessarily mean that *N* is always prime? If this were the case, then this theorem could rigorously be employed as a prime number sieve.

Unfortunately the converse is not always true, as is evidenced by the existence of the so-called Fermat pseudo-primes.

Why do these pseudo-primes exist? To answer this question, the concept of a relative prime will now be discussed, followed by Theorem 88 of Hardy and Wright. This will then pave the way to explain a simple method of eliminating the pseudo-primes, thus making it possible to indeed employ Fermat's Little Theorem in a prime number sieve.

Relative primes (or co-primes) and Φ(N)

A relative prime number (or co-prime) can be defined as any integer (excluding the number 1) below a given number *N* which is ***not a factor, or a multiple of a factor***, of that number.

The number of relative primes below *N* is denoted by the symbol *Φ(N).*

Take for example *N* = 21: 21 = 3 x 7, so both 3 and 7 are factors of 21.These factors and their multiples below 21 are: 3, 6, 9,12,15,18 as well as 7 and 14. There thus exist (7-1) = 6 multiples of 3, as well as (3-1) = 2 multiples of 7 below *N* = 21.

These ***eight*** integers are ***not relatively prime*** with regards to *N* = 21.

Denoting the number of ***non-relative primes*** by the symbol n_p we can write, for *N* = 21:

$$n_p = (3-1) + (7-1) = 2 + 6 = 8.$$

Because $$(N-1) = \Phi(N) + n_p$$

the number of co-primes below 21 must therefore be

$$\begin{aligned} \Phi(21) &= (21-1) - n_p \\ &= (21-1) - (3-1) - (7-1) \\ &= (21-1) - 8 \end{aligned}$$

Thus $$\Phi(21) = 12$$

The twelve relative primes below 21 are: 2, 4, 5, 7, 8, 10, 11, 13, 16, 17, 19 and 20.

Because a prime number has no factors other than 1 and itself, it follows that, if *N* is prime $n_p = 0$ and thus

$$\Phi(N) = (N-1) \quad \text{for a prime number}$$

In general, for $N = a^p \times b^q \times c^r \times \ldots$ where a, b, c etc. are prime factors of *N, Φ(N)* can be calculated using the formula

$$\phi(N) = N\left(1-\frac{1}{a}\right)\left(1-\frac{1}{b}\right)\left(1-\frac{1}{c}\right)\ldots$$

e.g. $$\phi(21) = 21\left(1-\frac{1}{3}\right)\left(1-\frac{1}{7}\right) = 21\left(\frac{2}{3}\right)\left(\frac{6}{7}\right) = 12$$

Because $$n_p = (N-1) - \Phi(N)$$

Thus $$n_p = (N-1) - N\left(1-\frac{1}{a}\right)\left(1-\frac{1}{b}\right)\left(1-\frac{1}{c}\right)\ldots$$

$$n_p = (N-1) - N\left(\frac{a-1}{a}\right)\left(\frac{b-1}{b}\right)\left(\frac{c-1}{c}\right)\ldots$$

Using this formula, it is possible to find many different formulations for n_p depending on the combination of prime factors belonging to a particular non-prime *N.* Some such formulas, related mostly to the ***pseudo-primes*** identified within the range of this investigation, are:

For *N* = *ab*: $$\begin{aligned} n_p &= (ab-1) - ab\left(\frac{a-1}{a}\right)\left(\frac{b-1}{b}\right) \\ &= (a-1) + (b-1) \end{aligned}$$

For $N = a^2$: $$\begin{aligned} n_p &= a^2 - 1 - a^2\left(\frac{a-1}{a}\right) \\ &= a - 1 \end{aligned}$$

For *N* = *abc:* $$\begin{aligned} n_p &= (abc-1) - abc\left(\frac{a-1}{a}\right)\left(\frac{b-1}{b}\right)\left(\frac{c-1}{c}\right) \\ &= a(b-1) + b(c-1) + c(a-1) \end{aligned}$$

For $N = a^2b$: $$n_p = (a^2b - 1) - a^2b\left(\frac{a-1}{a}\right)\left(\frac{b-1}{b}\right)$$
$$= \left(a^2 - 1\right) + a(b-1)$$

These formulas are employed to help explain the method for rooting out pseudo-primes.

Theorem 88 of Hardy and Wright (given without proof)

"$10^x = 1$ (mod N) has a smallest solution x which is a divisor of $\Phi(N)$"

where $\Phi(N)$ is equal to the number of integers smaller than N which are relatively prime.

Stated in another way:

If $\frac{10^x}{N}$ is calculated and a remainder equal to 1 is obtained (i.e. recurrence occurs), then the smallest possible value for x is a divisor of $\Phi(N)$.

This theorem is, in a sense, a converse of Fermat's Little Theorem (with base $a = 10$), except that the conclusion is not that N is necessarily prime, but rather that the smallest possible value for x (which represents the number of recurring decimals in the string for $\frac{1}{N}$) will always divide into $\Phi(N)$.

If $\Phi(N)$ happens to equal $N - 1$ (which only happens for a prime number), it follows that the smallest possible value for x is a divisor of $N - 1$. Thus for a prime, the number of recurring decimals in the string for $\frac{1}{N}$ will always be a divisor of N-1. This is Fermat's Little Theorem.

An example of applying Theorem 88 to a non-prime:

For $N = 21$ (with prime factors 3 and 7)

$$\frac{1}{21} = 0,\dot{0}4761\dot{9}$$

There are $x = 6$ recurring digits in the decimal string.

Thus $\frac{10^6}{21} = 47619 + \frac{1}{21}$ but also $\frac{10^{12}}{21} = 47\,619\,047\,619 + \frac{1}{21}$

and $\frac{10^{18}}{21} = 47\,619\,047\,619\,047\,619 + \frac{1}{21}$ etc.

For $N = 21$, $\Phi(21) = 12$. The smallest possible value of x (i.e. 6) is a divisor of $\Phi(21)$, as 12/6 = 2.

Theorem 88 can thus be used to demonstrate how the number of recurring digits x in $1/N$ is related to $\Phi(N)$, the number of integers smaller than N which are relatively prime:

- In the case of N being ***prime***, x can always be divided exactly into $N - 1$, because $\Phi(N) = N - 1$.

- However, in the case of N being a ***non-prime***, x is a divisor of $N - 1 - n_p$ (not necessarily of N - 1) because $\Phi(N) = N - 1 - n_p$.

Explanation of the existence of the pseudo-primes

This last point can be used to explain the existence of the "culprits" or Fermat pseudo-primes (base 10): Although x is always a divisor of *Φ(N),* it is usually not a divisor of n_p as well. Should the latter be true, then x would indeed divide exactly into (N – 1), because $N - 1 = \Phi(N) + n_p$. This happens to be the case for the "culprits".

Take, for example, the third smallest pseudo-prime 1/91

$\frac{1}{91} = 0.\dot{0}1098\dot{9}$ with x = 6 and $k = \frac{91-1}{6}$ = 15

91 = 7 x 13, and is thus not a prime number.

For 91, n_p = (7 – 1) + (13 – 1) = 6 + 12 = 18

$\Phi(91)$ = (91 – 1) – 18 = 72

Thus $\frac{N-1}{x} = \frac{\Phi(N)}{x} + \frac{n_p}{x}$

Thus $k = \frac{72}{6} + \frac{18}{6}$

$= 12 + 3$

$= 15$

Because n_p = 18 happens to be divisible by 6, the k-value for N = 91 is a whole number, resulting in 91 "slipping through" a potential prime number sieve.

Compare this with another non-prime that is ***not*** a "culprit":

For N = 21 (= 3 x 7), it has already been shown that x = 6 and n_p = 8. Here n_p is not divisible by 6. The k-value of N = 21 equals 20/6 = $3,\dot{3}$, thus $\frac{(N-1)}{x}$ is not an integer (although $\frac{\Phi(N)}{x}$ is).

A further simple test to eliminate "culprits" or "pseudo-primes"

Say a non-prime "culprit" has slipped through the first step of the Fermat sifting process, due to its N – 1 value being found to be divisible by x. This is, of course, due to its n_p value being divisible by x.

$$(N - 1) = \Phi(N) + np$$

and also $$\frac{(N-1)}{x} = \frac{\Phi(N)}{x} + \frac{n_p}{x}$$

For a pseudo-prime with prime factors: $N = ab$

$$n_p = (a - 1) + (b - 1)$$

$$\frac{n_p}{x} = \frac{(a-1)}{x} + \frac{(b-1)}{x}$$

If n_p is divisible by x, then there is very likely a term $(a - 1)$ also divisible by x.

Thus $\frac{(a-1)}{x} = d$ where d is an integer < k

It follows that at least one of the factors of the pseudo-prime can be written as a function of x:

$$a = dx + 1$$

Example: Say we wish to test whether N = 703 (with x = 18 and k = 39) is a real prime or not. (Remember: $k = \frac{(N-1)}{x}$ thus also $N = kx + 1 = (39)(18) + 1 = 703$)

Instead of subjecting 703 to a process of testing for divisibility by ***all primes*** smaller than itself until a factor is found (the conventional test for primeness) ***only possible factors which satisfy the condition a = dx + 1, need to be tested***. If the only d-value which satisfies this condition equals k itself, then a = N, and N is thus shown to be prime.

Test for d = 1: $a = dx + 1 = (1)(18) + 1 = 19$
Test if 19 is a factor: $\frac{703}{19} = 37$

Thus the "culprit" 703 is almost immediately shown to be non-prime.

(Note also: $dx + 1 = (2)(18) + 1 = 37$ quickly yields the other prime factor of 703 as well.)

For a pseudo-prime with prime factors: $N = a^2 b$

$$n_p = (a^2 - 1) + a(b-1)$$

$$\frac{n_p}{x} = \frac{(a^2-1)}{x} + \frac{a(b-1)}{x}$$

If n_p is divisible by x, then there is very likely a term $(a^2 - 1)$ also divisible by x.

Thus $\frac{(a^2-1)}{x} = d$ where d is an integer < k

A factor of the pseudo-prime can therefore be:

$$a^2 = dx + 1$$

Example, say we wish to test whether N = 1233 (with x = 8 and k = 154) is a real prime or not. (Remember: $k = \frac{(N-1)}{x}$ thus also $N = kx + 1 = (154)(8) + 1 = 1233$)

Test for a possible factor $a^2 = dx + 1$:

Test for d = 1: $a^2 = dx + 1 = (1)(8) + 1 = 9$
Test if 9 is a factor: $\frac{1233}{9} = 137$

Thus the "culprit" 1233 is almost immediately shown to be non-prime.

(Note also: $dx + 1 = (17)(8) + 1 = 137$ yields the other prime factor of 1233.)

For a pseudo-prime with prime factors: $N = abc$

$$n_p = a(b-1) + b(c-1) + c(a-1)$$

$$\frac{n_p}{x} = \frac{a(b-1)}{x} + \frac{b(c-1)}{x} + \frac{c(a-1)}{x}$$

A factor of the pseudo-prime is therefore likely to be in the form:

$$a = dx + 1$$

Two examples of testing for pseudo-primes $N = abc$

Test for $N = 561$ (with $x = 16$ and $k = 35$)

Test for a possible factor $a = dx + 1$:

Test for $d = 1$: $a = dx + 1 = (1)(16) + 1 = 17$

Test if 17 is a factor: $\frac{561}{17} = 33$

Thus the "culprit" 561 is again almost immediately shown to be non-prime.

(Note also: $dx + 1 = (2)(16) + 1 = 33$ yields another factor of 561. Thus 3 x 11 x 17 =561 =abc)

Test for $N = 8911$ (with $x = 198$ and $k = 45$)

In this particular case, no integer value of *d* yields a factor of 8911.

However d = $\frac{1}{3}$ yields a factor.

Explanation: 8911 = 7 x 19 x 67 with $n_p = 1782$

Using:

$$\frac{n_p}{x} = \frac{a(b-1)}{x} + \frac{b(c-1)}{x} + \frac{c(a-1)}{x}$$

$$\frac{1782}{198} = \frac{7(19-1)}{198} + \frac{19(67-1)}{198} + \frac{67(7-1)}{198}$$

$$\frac{1782}{198} = \frac{7(18)}{198} + \frac{19(66)}{198} + \frac{67(6)}{198}$$

$$9 = \frac{7}{11} + \frac{19(66)}{3(66)} + \frac{67}{33}$$

None of the terms on the RHS of this equation are divisible by x, although their sum is!

However, consider the term: $\frac{19(66)}{3(66)} = \frac{19(67-1)}{198}$

$$\frac{(67-1)}{198} = \frac{1}{3}$$

Thus: $67 = \frac{1}{3}(198) + 1$

Thus: $a = \frac{1}{3}(x) + 1$

Hence a factor in the form $a = dx + 1$ can indeed be found, although $d = \frac{1}{3}$ is not an integer.

For the 31 pseudo-primes below 10000, only four of them (among the larger ones) have non-integer *d values.* They required $d = \frac{1}{2}, \frac{1}{3}, \frac{1}{15}$ and $\frac{1}{17}$ respectively in order to identify a factor. (See Table 3.) Thus fractional *d*-values must also be considered in each step of the rooting out procedure.

Be that as it may, all the pseudo-primes below 10000 can indeed be successfully rooted out by this procedure. In the majority of cases, *d* equals 1 or 2, thus a factor is found almost immediately, and these *N*-values are identified very quickly as not being prime.

N values above 10000 were not studied. Obviously, larger pseudo-primes exist with far more complicated prime factorizations $a^p b^q c^r d^s$ .... etc. It is assumed that factors in the form *dx*+1 can be found for all such higher “culprits”.

It is of interest to note - refer to Tables 2 and 3 - that the x value of a pseudo-prime number is generally very much smaller than that of a real prime number. For large *N* values, it is therefore not unreasonable to suspect pseudo-prime status when *k* is indeed an integer but there are relatively few digits in the recurring decimal string for $\frac{1}{N}$.

Obviously, the testing for a factor $dx + 1$ must be done for ***all*** *N* values with integer *k*-values greater than 1 (thus primes as well as pseudo-primes). Although this may appear to involve a large amount of calculation steps, the work may be considerably less than expected; the reasons being:

(a) x values for real primes are usually quite large, thus, generally, relatively few *d* values need to be tested in order to reveal that no other factors exist but *N* = $kx + 1$ itself.

(b) *d* values need only be tested up until a possible factor $dx + 1$ which is not bigger than half of *N*’s value.

(c) It has been empirically observed (between 3 and 10000) that no pseudo-primes exist with *k* < 8. Thus it might only be necessary to subject *N*-values with $k \geq 8$ to the $dx + 1$ test.

(d) The amount of work required by the additional necessity to test for fractional *d*-values (if an integer *d* does not reveal a factor) is also not considerable, as the product of x and the fraction involved, must yield an integer.

Example: Test for *N* = 211 (a real prime with x = 30 and *k* = 7)

$$\left.\begin{aligned}
dx + 1 &= (1)(30) + 1 = 31 \\
dx + 1 &= (2)(30) + 1 = 61 \\
dx + 1 &= (3)(30) + 1 = 91 \\
dx + 1 &= (4)(30) + 1 = 120 \quad \text{Can stop here, as } 120 > \tfrac{1}{2}(241) \\
\\
dx + 1 &= \left(\frac{1}{2}\right)(30) + 1 = 16 \\
dx + 1 &= \left(\frac{1}{3}\right)(30) + 1 = 11 \\
dx + 1 &= \left(\frac{1}{5}\right)(30) + 1 = 7 \\
dx + 1 &= \left(\frac{1}{6}\right)(30) + 1 = 6 \\
dx + 1 &= \left(\frac{1}{10}\right)(30) + 1 = 4 \\
dx + 1 &= \left(\frac{1}{15}\right)(30) + 1 = 3 \quad \text{Can stop here, as 3 is smallest possible factor.}
\end{aligned}\right\} \text{No factors found}$$

A summary of the steps in the prime number test:

1) Only test for *N*-values ending on 1, 3, 7 or 9.

2) Calculate x, the number of recurring digits in the decimal string for $\frac{1}{N}$.

3) Calculate the value *of k* = $\frac{N-1}{x}$. If *k* = 1 then *N* is prime, and if *k* is not a whole number then *N* is not prime. However, if *k* is a whole number greater than 1*, one more step is required:

4) Inspect whether a value dx + 1 exists which divides exactly into *N*. If a *d*-value smaller than *k* generates a factor of *N*, then *N* is not prime. This step appears to successfully root out all pseudo-primes.

(It might only be necessary to test k-values > 8, according to the results of this analysis.)*

Conclusion

1) By applying the four steps summarized above, all prime numbers between 3 and 10000 (excluding 2 and 5) were successfully distinguished from non-primes.

2) From this mainly empirical study it appears that, in the case of an ***even*** number of digits in the recurring decimal string for $\frac{1}{N}$, if the first half of the string is added to the second half, and a string of 9's is ***not*** obtained, *N* is always ***non-prime***. Primes always generate strings of 9's, while some non-primes do as well.

3) Furthermore, in the case of an ***odd*** number of digits in the recurring decimal string for $\frac{1}{N}$, when the ***digital root*** (*DR*) of all the numbers in one cycle of the recurring string does ***not*** equal 9, *N* is always ***non-prime***. All primes were found to have *DR* values equal to 9, but some non-primes as well.

4) The findings stated in points 2 and 3 above might also be used to help sift non-primes from primes. See the appendix for a more detailed discussion.

5) This investigation involved the application of three Vedic Mathematics techniques, namely the use of the two sutras ***Ekadhikena Purvena*** and ***Nikhilam Navatascaramam Dasatah*** as well as the use of ***digital roots***. Vedic mathematics can therefore be employed as a powerful tool to help penetrate and obtain a deeper understanding of various aspects of number theory.

APPENDIX

Some additional comments:

<u>Note 1</u>

The phenomenon of the addition of the first half of an even digit string to its second half very often yielding a string of 9's (application of Nikhilam Navatascaramam Dasatah)

Multiples of 1/7:	1/7	=	0.142857 …
	2/7	=	0.285714 …
	3/7	=	0.428571 …
	4/7	=	0.571428 …
	5/7	=	0.714285 …
	6/7	=	0.857142 …

For this prime number, the decimal strings of twice $\frac{1}{N}$, three times $\frac{1}{N}$ … up to (*N*-1) times $\frac{1}{N}$, all consist of the ***same sequence of numbers***, just starting at different digits in the sequence. Because 6 different multiples of $\frac{1}{7}$ all yield the same sequence, a minimum of six recurring digits is sufficient to accommodate each multiple starting at a different digit. For *N* = 7, x is thus 6.

Furthermore $\frac{1}{7}+\frac{6}{7}=\frac{7}{7}=1$ It therefore follows that

$$\begin{array}{r} 0.142857 \\ +\ \underline{0.857142} \\ \underline{0.999999} \end{array}$$

Similarly $\frac{2}{7}+\frac{5}{7}=\frac{7}{7}=1$ It therefore follows that

$$\begin{array}{r} 0.285714 \\ +\ \underline{0.714285} \\ \underline{0.999999} \end{array}$$

Similarly $\frac{3}{7}+\frac{4}{7}=\frac{7}{7}=1$ It therefore follows that

$$\begin{array}{r} 0.428571 \\ +\ \underline{0.571428} \\ \underline{0.999999} \end{array}$$

Thus, in the case for $\frac{1}{7}$, for "n" a number between 1 and 6, the first half of the string for $\frac{n}{7}$ is identical to the second half of the string for $\frac{7-n}{7}$, while the second half of the string for $\frac{n}{7}$ is identical to the first half of the string for $\frac{7-n}{7}$. Because $\frac{n}{7}+\frac{7-n}{7}$ must equal 1 (which is the limit towards which $.\dot{9}$ tends), the first half and the second half of any string must add to only digits of 9's .

This phenomenon can only occur for a cyclical string with an ***even*** number of digits, as such a string can be divided exactly into halves.

The generation of the second half of the string from the first by subtracting each digit from 9 is an application of the ***Nikhilam Navatascaramam Dasatah*** sutra (all from 9 and the last from 10). Because the string is non-terminating "the last" digit never occurs, and thus no subtraction from 10 occurs.

The process whereby the second half is generated from the first (according to ***Nikhilam Navatascaramam Dasatah***) can be shown by considering the steps in a conventional long division process:

$\frac{1}{7} = \left(\frac{10}{7}\right)(0{,}1)$
$\frac{1}{7} = \left(1 + \frac{3}{7}\right)(0{,}1)$
$\frac{1}{7} = 1(0{,}1) + \left(\frac{30}{7}\right)(0{,}1)^2$
$\frac{1}{7} = 1(0{,}1) + \left(4 + \frac{2}{7}\right)(0{,}1)^2$
$\frac{1}{7} = 1(0{,}1) + 4(0{,}1)^2 + \left(\frac{20}{7}\right)(0{,}1)^3$
$\frac{1}{7} = 1(0{,}1) + 4(0{,}1)^2 + \left(2 + \frac{6}{7}\right)(0{,}1)^3$
$\frac{1}{7} = 1(0{,}1) + 4(0{,}1)^2 + 2(0{,}1)^3 + \left(\frac{6}{7}\right)(0{,}1)^3$
$\frac{1}{7} = 1(0{,}1) + 4(0{,}1)^2 + 2(0{,}1)^3 + \left(1 - \frac{1}{7}\right)(0{,}1)^3$ …Equation 1

But $\left(1 - \frac{1}{7}\right) = 0.999 + 0{,}1^3 - \frac{1}{7}$
$= 0.9 + 0{,}09 + 0{,}009 - (\frac{1}{7}) + 0{,}1^3$

Substitute Equation 1 into the term $(\frac{1}{7})$ above.

$$\left(1 - \frac{1}{7}\right) = 9(0{,}1)^1 + 9(0{,}1)^2 + 9(0{,}1)^3 - \left[1(0{,}1)^1 + 4(0{,}1)^2 + 2(0{,}1)^3 + \left(1 - \frac{1}{7}\right)(0{,}1)^3\right] + 0{,}1^3$$

Rearranging terms and regrouping yields

$\left(1 - \frac{1}{7}\right) = (9-1)(0{,}1)^1 + (9{-}4)(0{,}1)^2 + (9{-}2)(0{,}1)^3 - \left(1 - \frac{1}{7}\right)(0{,}1)^3 + 0{,}1^3$
$= (9-1)(0{,}1)^1 + (9{-}4)(0{,}1)^2 + (9{-}2)(0{,}1)^3 - 0{,}1^3 + 0{,}1^3 + \frac{1}{7}(0{,}1)^3$ …Equation 2
$= (8)(0{,}1)^1 + (5)(0{,}1)^2 + (7)(0{,}1)^3 + \frac{1}{7}(0{,}1)^3$ …Equation 3

Substitute Equation 3 into Equation 1

$\frac{1}{7} = 1(0{,}1) + 4(0{,}1)^2 + 2(0{,}1)^3 + \left[8(0{,}1)^1 + 5(0{,}1)^2 + 7(0{,}1)^3 + \frac{1}{7}(0{,}1)^3\right](0{,}1)^3$
$\frac{1}{7} = 1(0{,}1) + 4(0{,}1)^2 + 2(0{,}1)^3 + 8(0{,}1)^4 + 5(0{,}1)^5 + 7(0{,}1)^6 + \frac{1}{7}(0{,}1)^7$
$\frac{1}{7} = 0{,}142857 + \frac{1}{7}(0{,}1)^7$

Equation 2 clearly shows how the digits in the second half of the string are generated from the digits in the first half of the string via repeated subtractions from 9.

$\frac{1}{19}$ ***displays exactly the same phenomenon:***

Multiples of 1/19:					
	1/19	=	0. 0 5 2 6 3 1 5 7 8	9 4 7 3 6 8 4 2 1	…
	2/19	=	0. 1 0 5 2 6 3 1 5 7	8 9 4 7 3 6 8 4 2	…
	3/19	=	0. 1 5 7 8 9 4 7 3 6	8 4 2 1 0 5 2 6 3	…
	4/19	=	0. 2 1 0 5 2 6 3 1 5	7 8 9 4 7 3 6 8 4	…
	5/19	=	0. 2 6 3 1 5 7 8 9 4	7 3 6 8 4 2 1 0 5	…
	6/19	=	0. 3 1 5 7 8 9 4 7 3	6 8 4 2 1 0 5 2 6	…
	7/19	=	0. 3 6 8 4 2 1 0 5 2	6 3 1 5 7 8 9 4 7	…
	8/19	=	0. 4 2 1 0 5 2 6 3 1	5 7 8 9 4 7 3 6 8	…
	9/19	=	0. 4 7 3 6 8 4 2 1 0	5 2 6 3 1 5 7 8 9	…
	10/19	=	0. 5 2 6 3 1 5 7 8 9	4 7 3 6 8 4 2 1 0	…
	11/19	=	0. 5 7 8 9 4 7 3 6 8	4 2 1 0 5 2 6 3 1	…
	12/19	=	0. 6 3 1 5 7 8 9 4 7	3 6 8 4 2 1 0 5 2	…
	13/19	=	0. 6 8 4 2 1 0 5 2 6	3 1 5 7 8 9 4 7 3	…
	14/19	=	0. 7 3 6 8 4 2 1 0 5	2 6 3 1 5 7 8 9 4	…
	15/19	=	0. 7 8 9 4 7 3 6 8 4	2 1 0 5 2 6 3 1 5	…
	16/19	=	0. 8 4 2 1 0 5 2 6 3	1 5 7 8 9 4 7 3 6	…
	17/19	=	0. 8 9 4 7 3 6 8 4 2	1 0 5 2 6 3 1 5 7	…
	18/19	=	0. 9 4 7 3 6 8 4 2 1	0 5 2 6 3 1 5 7 8	…

Because 18 different multiples of $\frac{1}{19}$ all yield the same sequence, a minimum of 18 recurring digits is sufficient to accommodate each multiple starting at a different digit in the string. In this case the first half of the string for $\frac{n}{19}$ is identical to the second half of the string for $\frac{19-n}{19}$, while the second half of the string for $\frac{n}{19}$ is identical to the first half of the string for $\frac{19-n}{19}$.

Thus $\frac{1}{19}+\frac{18}{19}=\frac{19}{19}=1$; $\frac{2}{19}+\frac{17}{19}=\frac{19}{19}=1$; $\frac{3}{19}+\frac{16}{19}=\frac{19}{19}=1$ etc.

$\frac{1}{19}+\frac{18}{19}$ = 0. 0 5 2 6 3 1 5 7 8 9 4 7 3 6 8 4 2 1 +

0. 9 4 7 3 6 8 4 2 1 0 5 2 6 3 1 5 7 8

0. 9 9 9 9 9 9 9 9 9 9 9 9 9 9 9 9 9 9 etc.

A case where k=2

For both $\frac{1}{N}$ = $\frac{1}{7}$ as well as $\frac{1}{N}$ = $\frac{1}{19}$ the number of recurring decimals divide exactly into *N*-1, i.e. $k=\frac{N-1}{x}=1$. However, for the prime number 13, $k=\frac{13-1}{6}=2$, as $\frac{1}{13}$ only has x = 6 recurring decimals. Why is this so?

Multiples of 1/13:

1/13	=	0.076923	A
2/13	=	0.153846	B
3/13	=	0.230769	A
4/13	=	0.307692	A
5/13	=	0.384615	B
6/13	=	0.461538	B
7/13	=	0.538461	B
8/13	=	0.615384	B
9/13	=	0.692307	A
10/13	=	0.769230	A
11/13	=	0.846153	B
12/13	=	0.923076	A

In this case, $\frac{2}{13}$ does not yield the same sequence of numbers in its decimal string as $\frac{1}{13}$. Calling the sequence of numbers belonging to $\frac{1}{13}$ "family A" and the sequence of numbers belonging to $\frac{2}{13}$ "family B", it is possible to group all the multiples of $\frac{1}{13}$ (up to $\frac{12}{13}$) into either family A or family B. Within each family the digits occur in exactly the same order, just starting at different points in the sequence. It is only required that there be ***six*** digits in each string: 6 digits in string A accommodating starting points for 6 out of the 12 multiples of 13 which are common fractions, and likewise 6 digits in string B accommodating starting points for the other six multiples of $\frac{1}{13}$.

Note the symmetry in the distribution of families listed in order of increasing multiples:

A B A A B B B B A A B A

The ***Nikhilam Navatascaramam Dasatah*** sutra still applies here, as there will always be pairs of fractions (each pair belonging to the same family A or B), such that $\frac{n}{13}+\frac{13-n}{13}$ =1 (or $\mathbf{0.\dot{9}}$), i.e.

1/13 + 12/13 = 1 0.076923 A

0.923076 A

0.999999

Also: 5/13 + 8/13 = 1 0.384615 B

0.615384 B

0.999999 etc.

Note 2:

A case where strings of 3's and 6's are obtained when the first half of a string is added to the second half of a string

Multiples of 1/21: (21 is a non-prime with factors 3 and 7)

1/21	=	0.047619	A	
2/21	=	0.095238	B	
3/21	**=**	**0.142857**	**=**	**1/7**
4/21	=	0.190476	A	
5/21	=	0.238095	B	
6/21	**=**	**0.285714**	**=**	**2/7**
7/21	**=**	**0.333...**	**=**	**1/3**
8/21	=	0.380952	B	
9/21	**=**	**0.428571**	**=**	**3/7**
10/21	=	0.476190	A	
11/21	=	0.523809	B	
12/21	**=**	**0.571428**	**=**	**4/7**
13/21	=	0.619046	A	
14/21	**=**	**0.666...**	**=**	**2/3**
15/21	**=**	**0.714285**	**=**	**5/7**
16/21	=	0.761904	A	
17/21	=	0.809523	B	
18/21	**=**	**0.857142**	**=**	**6/7**
19/21	=	0.904761	A	
20/21	=	0.952380	B	

In this case, $\frac{2}{21}$ does not yield the same sequence of numbers in its decimal string as $\frac{1}{21}$. Calling the sequence of numbers belonging to $\frac{1}{21}$ "family A" and the sequence of numbers belonging to $\frac{2}{21}$ "family B", it is possible to group ***twelve*** of the multiples of $\frac{1}{21}$ into either family A or family B. Within each family the digits occur in exactly the same order, just starting at different points in the sequence.

Furthermore, multiples of $\frac{1}{21}$ which belong to either "family", correspond to numbers which are ***relatively prime*** with regards to N = 21, i.e.

1, 2, 4, 5, 8, 10, 11, 13, 16, 17, 19 and 20 totalling $\Phi(21) = 12$.

It is only required that there be ***six*** digits in each string: 6 digits in string A accommodating starting points for 6 out of the 12 relatively prime multiples of $\frac{1}{21}$, and likewise 6 digits in string B accommodating starting points for the other six relatively prime multiples of $\frac{1}{21}$. Thus x = 6 for the relatively prime multiples of $\frac{1}{21}$.

The other multiples of $\frac{1}{21}$, can be reduced to simpler fractions; they are thus ***prime factor multiples*** of $\frac{1}{21}$, and can be reduced to multiples of $\frac{1}{3}$ and $\frac{1}{7}$. These multiples of $\frac{1}{21}$ correspond to numbers which are ***not relatively prime*** with respect to 21, i.e. 3, 6, 9,12,15,18, totalling 6; as well as 7 and 14, totalling 2. Thus n_p = 6 + 2 = 8.

Furthermore, addition of the first half of each string to its second half, does not, in this case, yield a string of 9's. A more general rule now appears:

The first half can add to the second half to yield the string of digits belonging to the reciprocal of a factor (or a multiple of the reciprocal of the factor) of N, in this case 3 is a factor of 21.

Family A: $\frac{1}{21} = 0.\dot{0}4761\dot{9}$ $\frac{13}{21} = 0.61904\dot{7}$ $\frac{1}{21} + \frac{13}{21} = \frac{14}{21} = \frac{2\times7}{3\times7} = \frac{2}{3}$ = 0.666…

$$\begin{array}{l} 047 \\ \underline{619} \\ \underline{666} \end{array} \qquad \begin{array}{l} 619 \\ \underline{047} \\ \underline{666} \end{array}$$

Family B: $\frac{2}{21} = 0.\dot{0}9523\dot{8}$ $\frac{5}{21} = 0.\dot{2}3809\dot{5}$ $\frac{2}{21} + \frac{5}{21} = \frac{7}{21} = \frac{1\times7}{3\times7} = \frac{1}{3}$ = 0.333…

$$\begin{array}{l} 095 \\ \underline{238} \\ \underline{333} \end{array} \qquad \begin{array}{l} 238 \\ \underline{950} \\ \underline{333} \end{array}$$

For $\frac{1}{21}$ the process whereby the second half of the string is generated from the first (which is similar to applying ***Nikhilam Navatascaramam*** except that digits are now subtracted from 6 and not from 9) can be demonstrated by considering the steps in the conventional long division process:

$$\frac{1}{21} = \left(\frac{10}{21}\right)(0{,}1)$$
$$\frac{1}{21} = \left(0 + \frac{10}{21}\right)(0{,}1)$$
$$\frac{1}{21} = 0(0{,}1) + \left(\frac{100}{21}\right)(0{,}1)^2$$
$$\frac{1}{21} = 0(0{,}1) + \left(4 + \frac{16}{21}\right)(0{,}1)^2$$
$$\frac{1}{21} = 0(0{,}1) + 4(0{,}1)^2 + \left(\frac{160}{21}\right)(0{,}1)^3$$
$$\frac{1}{21} = 0(0{,}1) + 4(0{,}1)^2 + \left(7 + \frac{13}{21}\right)(0{,}1)^3$$
$$\frac{1}{21} = 0(0{,}1) + 4(0{,}1)^2 + 7(0{,}1)^3 + \left(\frac{13}{21}\right)(0{,}1)^3$$
$$\frac{1}{21} = 0(0{,}1) + 4(0{,}1)^2 + 7(0{,}1)^3 + \left(\frac{14}{21} - \frac{1}{21}\right)(0{,}1)^3 \qquad \text{…Equation 1}$$

But $\left(\frac{14}{21} - \frac{1}{21}\right) = \left(\frac{2}{3} - \frac{1}{21}\right) = 0.666 + \left(\frac{2}{3}\right)(0{,}1)^3 - \frac{1}{21} = 0.6 + 0{,}06 + 0{,}006 - (\frac{1}{21}) + \left(\frac{2}{3}\right)0{,}1^3$

Substitute Equation 1 into the term $(\frac{1}{21})$ above.

$$\left(\frac{14}{21}-\frac{1}{21}\right)= 6(0{,}1)^1 + 6(0{,}1)^2 + 6(0{,}1)^3 - \left[0(0{,}1)^1 + 4(0{,}1)^2 + 7(0{,}1)^3 + \left(\frac{14}{21}-\frac{1}{21}\right)(0{,}1)^3\right] + \left(\frac{2}{3}\right)0{,}1^3$$

Rearranging terms and regrouping yields

$$\left(\frac{14}{21}-\frac{1}{21}\right) = (6-0)(0{,}1)^1 + (6-4)(0{,}1)^2 + (6-7)(0{,}1)^3 - \left(\frac{2}{3}-\frac{1}{21}\right)(0{,}1)^3 + \left(\frac{2}{3}\right)0{,}1^3 \quad \text{...Equation 2}$$

Carrying over a 10's digit from the second to the third term yields

$$\left(\frac{14}{21}-\frac{1}{21}\right) = (6-0)(0{,}1)^1 + (5-4)(0{,}1)^2 + (16-7)(0{,}1)^3 - \left(\frac{2}{3}\right)0{,}1^3 + \left(\frac{2}{3}\right)0{,}1^3 + \frac{1}{21}(0{,}1)^3$$
$$= 6(0{,}1)^1 + (1)(0{,}1)^2 + (9)(0{,}1)^3 + \frac{1}{21}(0{,}1)^3 \quad \text{...Equation 3}$$

Substitute Equation 3 into Equation 1

$$\frac{1}{21} = 0(0{,}1) + 4(0{,}1)^2 + 7(0{,}1)^3 + \left[6(0{,}1)^1 + 1(0{,}1)^2 + 9(0{,}1)^3 + \frac{1}{21}(0{,}1)^3\right](0{,}1)^3$$
$$\frac{1}{21} = 0(0{,}1) + 4(0{,}1)^2 + 7(0{,}1)^3 + 6(0{,}1)^4 + 1(0{,}1)^5 + 9(0{,}1)^6 + \frac{1}{21}(0{,}1)^7$$
$$\frac{1}{21} = 0{,}047619 + \frac{1}{21}(0{,}1)^7$$

Equation 2 clearly shows how the digits in the second half of the string are generated from the digits in the first half of the string via subtraction from a string of 6's.

Another example:

As was observed earlier on, the decimal string for $\frac{1}{119}$ is an example of the two halves of a string adding to give the ***reciprocal of a factor 7*****:** (The factors of 119 are 17 and 7.)

$$\frac{1}{119} = 0.\dot{0}08403361344537815126050420168067226890756302521\dot{1}$$

where

	008403361344537815126050 +
	420168067226890756302521
	428571428571428571428571

Note 3:

Why, for a PRIME, the digital root (DR) of all the digits in a string with an ODD number of digits always equals 9; and why this is not always the case for a NON-PRIME (Refer to Table 4)

Example: Some multiples of 1/53 (prime) are:

Family A	$\frac{1}{53} = 0{,}0\,1\,8\,8\,6\,7\,9\,2\,4\,5\,2\,8\,3$ ...	Family C	$\frac{52}{53} = 0{,}9\,8\,1\,1\,3\,2\,0\,7\,5\,4\,7\,1\,6$ ...
Family B	$\frac{2}{53} = 0{,}0\,3\,7\,7\,3\,5\,8\,4\,9\,0\,5\,6\,6$ ...	Family D	$\frac{51}{53} = 0{,}9\,6\,2\,2\,6\,4\,1\,5\,0\,9\,4\,3\,3$ ...
Family B	$\frac{3}{53} = 0{,}0\,5\,6\,6\,0\,3\,7\,7\,3\,5\,8\,4\,9$ ...	Family D	$\frac{50}{53} = 0{,}9\,4\,3\,3\,9\,6\,2\,2\,6\,4\,1\,5\,0$ ...
Family C	$\frac{4}{53} = 0{,}0\,7\,5\,4\,7\,1\,6\,9\,8\,1\,1\,3\,2$ ...	Family A	$\frac{49}{53} = 0{,}9\,2\,4\,5\,2\,8\,3\,0\,1\,8\,8\,6\,7$ ...
Family D	$\frac{5}{53} = 0{,}0\,9\,4\,3\,3\,9\,6\,2\,2\,6\,4\,1\,5$ ...	Family B	$\frac{48}{53} = 0{,}9\,0\,5\,6\,6\,0\,3\,7\,7\,3\,5\,8\,4$ ...
Family C	$\frac{6}{53} = 0{,}1\,1\,3\,2\,0\,7\,5\,4\,7\,1\,6\,9\,8$ ...	Family A	$\frac{47}{53} = 0{,}8\,8\,6\,7\,9\,2\,4\,5\,2\,8\,3\,0\,1$ ... etc.

1/53 has 4 “families” of 13 (uneven) digit strings, thus it has 4 x 13 = 52 strings, each starting at a different position in the sequence of numbers belonging to a particular family.

For $\frac{1}{53} + \frac{52}{53} = 1$

thus
$0\,,0\,1\,8\,8\,6\,7\,9\,2\,4\,5\,2\,8\,3\,\ldots$ *DR* = 9
$\underline{0\,,9\,8\,1\,1\,3\,2\,0\,7\,5\,4\,7\,1\,6\,\ldots}$ *DR* = 9
$\underline{0\,,9\,9\,9\,9\,9\,9\,9\,9\,9\,9\,9\,9\,9\,\ldots}$ *DR* = 9

Because these two sets of multiples of 1/53 must always add together to give a string of thirteen 9’s, and the digital root of these thirteen 9’s must itself equal 9, it appears that the digital root of each complementary string itself must also be 9. This occurs for any two sets of multiples which are “complements” in the sense that their sum equals one.

Thus it appears that, for a PRIME, the digital root (*DR*) of all the digits in a recurring string with an odd number of digits, always equals 9.

Compare the above ***prime*** denominator fraction 1/53 (with an uneven number of digits and *DR* = 9 for the digits in the string) to a ***non-prime*** denominator fraction, also with an uneven number of digits in a string, but with DR = 1, e.g. 1/81:

$\frac{1}{81} = 0\,,0\,1\,2\,3\,4\,5\,6\,7\,9\,\ldots \qquad \frac{80}{81} = 0\,,9\,8\,7\,6\,5\,4\,3\,2\,0\,\ldots$

Below 81(= 3^4) there are n_p = $3^3 - 1$ = 26 numbers which are not relatively prime, thus 81 – 1 – 26 = 54 (or $\Phi(81) = 3^4(3 - 1)/3 = 54$) co-primes. 1/81 has 9 digits in a recurring decimal string. Therefore there are 54 multiples of 1/81belonging to 54/9 = 6 “families” of strings.

Again, $\frac{1}{81} + \frac{80}{81} = 1$ Thus one of the families A: $0\,,0\,1\,2\,3\,4\,5\,6\,7\,9$ *DR* = 1
Another family B: $\underline{0\,,9\,8\,7\,6\,5\,4\,3\,2\,0}$ *DR* = 8
$\underline{0\,,9\,9\,9\,9\,9\,9\,9\,9\,9}$ *DR* = 9

Also, $\frac{2}{81} + \frac{79}{81} = 1$ Thus another family C: $0\,,0\,2\,4\,6\,9\,1\,3\,5\,8$ *DR* = 2
Another family D: $\underline{0\,,9\,7\,5\,3\,0\,8\,6\,4\,1}$ *DR* = 7
$\underline{0\,,9\,9\,9\,9\,9\,9\,9\,9\,9}$ *DR* = 9

Also, $\frac{4}{81} + \frac{77}{81} = 1$ Thus another family E: $0\,,0\,4\,9\,3\,8\,2\,7\,1\,6$ *DR* = 3
Another family F: $\underline{0\,,9\,5\,0\,6\,1\,7\,2\,8\,3}$ *DR* = 6
$\underline{0\,,9\,9\,9\,9\,9\,9\,9\,9\,9}$ *DR* = 9

Because the recurring patterns of any two sets of multiples of 1/53 which add together to make “1”, must always add together to give a string of nine 9’s, it appears that, if the digital root of one of the two complementary strings is *n*, then the digital root of the other string must be *9 - n*. This occurs for any two sets of multiples which are “complements” in the sense that their sum equals one.

Thus it appears that for a NON-PRIME denominator, the digital root (*DR*) of all the digits in a string with an odd number of digits, need not always equal 9, as long as the sum of its digital root with that of it’s complementary multiple still equals 9.

Table 4: All the Digital Roots (DR) of Cyclic 1/N Strings Containing an Odd Number of Digits for N values between 3 and 813

N	p or n	x	k	DR		N	p or n	x	k	DR
3	p	1	2	3		347	p	173	2	9
9**	n	1	8	1		359	p	179	2	9
27	n	3	8.66..	1		369	n	5	73.6	1
31	p	15	2	9		387	n	21	18.38..	3
37	p	3	12	9		397	p	99	4	9
41	p	5	8	9		431	p	215	2	9
43	p	21	2	9		439	p	219	2	9
53	p	13	4	9		443	p	221	2	9
67	p	33	2	9		453	n	75	6.026..	9
71	p	35	2	9		467	p	233	2	9
79	p	13	6	9		477	n	13	36.61..	5
81	n	9	8.88..	1		479	p	239	2	9
83	p	41	2	9		489	n	81	6.024..	9
93	n	15	6.133..	9		519	n	43	12.04..	6
107	p	53	2	9		523	p	261	2	9
111	n	3	36.66..	9		547	p	91	6	9
123	n	5	24.4	3		563	p	281	2	9
129	n	21	6.095..	9		573	n	95	6.021..	3
151	p	75	2	9		587	p	293	2	9
159	n	13	12.15..	6		597	n	99	6.020..	9
163	p	81	2	9		599	p	299	2	9
173	p	43	4	9		603	n	33	18.24..	6
191	p	95	2	9		613	p	51	12	9
199	p	99	2	9		631	p	315	2	9
201	n	33	6.060..	9		639	n	35	18.22..	9
213	n	35	6.057..	3		643	p	107	6	9
227	p	113	2	9		681	n	113	6.017..	1
237	n	13	18.15..	3		683	p	341	2	9
239	p	7	34	9		711	n	13	54.61..	7
243	n	27	8.96..	1		717	n	7	102.28..	6
249	n	41	6.048..	3		719	p	359	2	9
271	p	5	54	9		729	n	81	8.987..	1
277	p	69	4	9		733	p	61	12	9
279	n	15	18.53..	6		747	n	41	18.19..	7
283	p	141	2	9		751	p	125	6	9
307	p	153	2	9		757	p	27	28	9
311	p	155	2	9		773	p	193	4	9
317	p	79	4	9		787	p	393	2	9
321	n	53	6.037..	3		797	p	199	4	9
333	n	3	110.6..	3		813	n	5	162.4	6