The Magic of the last Digit

This paper is a development of Chapter 35 on Cube roots of Exact cubes explained in Bharati Krishna Tirthaji's book "Vedic Mathematics" published by Motilal Barnarsidass 1965.

At the commencement of the chapter, Tirthaji states that his method uses the Vilokanam Sutra (Observation) and also "Argumentation". No sutra is given for the latter, but I suggest the Sanskrit word Vitarka is appropriate. (Monier Williams in his Sanskrit dictionary on page 962 gives "Vitarka" as "reasoning, deliberation, consideration".)

It will be useful to begin by summarising the method developed by Tirthaji for extracting the cube root of an exact cube.

Since the cubes of the first 9 digits are : 1, 8, 27, 64, 125, 216, 343, 512, and 729 respectively, it will be observed that the final digit in each case is unique. Therefore the last digit of a cube root of an exact cube is obvious as seen in this table:

Last digit of the exact cube	Last digit of the cube root
1	1
2	8
3	7
4	4
5	5
6	6
7	3
8	2
9	9

Table 1: Relationship between the final digit of an exact cube and its cube root

To summarise:

- 1, 4, 5, 6, 9, and 0 repeat themselves in the cube endings
- 2 and 8, 3 and 7 interchange as complements from 10.

So to extract the cube root of say 389 017, we first separate out the digits from the right into groups of 3 as so: 389 / 017 (the need for this will be appreciated since 10^3 is 1000 and no cube of a single digit can exceed this). Since 017 ends in 7, the final digit of the cube root must be 3. (see chart above).

Looking now at the first group of three digits, namely 389, we notice it lies between the cube of 7 (i.e. 343) and the cube of 8(i.e. 512) and so the first digit of the cube root must be 7.

So the cube root of the exact cube 389 017 is 73.

A couple more illustrations may be helpful.

From the above, the final digit of cube root of 274 625 must be 5 since the last digit is 5. By Vilokanam, the first digit of the cube root must be 6 since 274 lies between 6^3 (216) and 7^3 (343). So the cube root of 274 625 is 65.

Likewise by using the same method, the cube root of 658 503 is 87.

Tirthaji proceeds to show how the cube roots of exact cubes of three and four digit numbers may be extracted by extending this process in a very simple and obvious way which is not gone into in this paper.

The purpose of this paper, however, is to show how a similar method can be applied to higher powers than cubes and their associated roots.

Now considering the fifth powers of the nine digits, we have:

Digit	Fifth Power	Approximation
1	1	1
2	32	30
3	243	250
4	1 024	1 000
5	3 125	3 000
6	7 776	8 000
7	16 807	17 000
8	32 768	33 000
9	59 049	59 000

Table 2:The fifth powers of the nine digits and their approximations

We will notice that the last digit of fifth powers are even more clearly related to the original digit than in the case of the cubes. In fact, they are identical! So 4 raised to the fifth power ends in 4, 7 raised to the fifth power ends in 7 etc.

To extract the fifth root of an exact power of a two digit number, we therefore proceed as for the cube root except in this case we take groups of five digits and not three. This is because 10^5 is 100 000 and no fifth power of a single digit can exceed 100 000.

Considering the example of extracting the fifth root of 147008443, we begin by dividing into groups of 5 digits from the right. i.e. 1470/08443. The leading group can have fewer than five digits, as seen in this case.

Since the last digit (i.e. the units digit) of the final group is 3 then the last digit of the fifth root must also be 3 by Vilokanam. Since the first group (1470) lies between 1024 and 3125 or, using the approximations which are simpler (column 3 of the above table), between 1000 and 3000 then, by argumentation, the first digit of the fifth root must be 4 so the fifth root of 147008443 is 43.

Likewise, the fifth root of 380204032 is 52 and that of 4704270176 is 86.

In the same way, the roots of exact powers of 7th, 9th and higher odd powers can be extracted and it is left to readers to develop this for themselves. Extraction of the roots of even powers will be considered later.

It is now instructive to consider the last digit of higher powers of the digits 1 to 9. These are listed in the table 3 below:

x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9	x^{10}	x^{11}	x^{12}	x^{13}	x^{14}	x^{15}	x^{16}	x^{17}
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	4	8	6	2	4	8	6	2	4	8	6	2	4	8	6	2
3	9	7	1	3	9	7	1	3	9	7	1	3	9	7	1	3
4	6	4	6	4	6	4	6	4	6	4	6	4	6	4	6	4
5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5
6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6
7	9	3	1	7	9	3	1	7	9	3	1	7	9	3	1	7
8	4	2	6	8	4	2	6	8	4	2	6	8	4	2	6	8
9	1	9	1	9	1	9	1	9	1	9	1	9	1	9	1	9

Table3: The final digits of the first 17 powers of the nine digits X = 1 to 9

Now, it will be observed that there is a clear repeating pattern of the final digit for every fourth power are the same. For instance, the final digits of 1st, 5th 9th 13th etc. powers (coloured red) are all 1, 2, 3, 4, 5, 6, 7, 8, 9 successively. In a similar way, the final digits 2nd, 6th, 10th, 14th etc. powers (coloured blue) are all 1, 4, 9, 6, 5, 6, 9, 4, 1 successively and likewise the final digits of the 3rd, 7th,11th, 15th powers are the same as are the final digits of the 4th, 8th, 12th, 16th powers. The question naturally arises as to why these patterns repeat every fourth power.

To understand the reason, we need to consider the multipliers that are required to move from the nth power to the (n+4)th power. These multipliers are the 4th power. For instance $3^1 \times 3^4 = 3^5$ and so the multiplier is 3^4. Since we are only concerned with the final digit there must be something significant in the final digits of the fourth powers which, when multiplying the 1st power, for instance, to keep them the same thus giving the same last digit for the 5th power. In other words these multipliers provide an "identity element" under multiplication for the final digit.

Some examples to illustrate the point will be useful:

8^1 when multiplied by the last digit of 8^4 (i.e. 6) gives a product 48 whose last digit is 8. Similarly, when 7^1 is multiplied by 7^4 whose final digit is 1, the final digit of the product will be 7.

Obviously, there is something significant in the last digit of the multipliers (i.e. fourth powers), which allow the final digit to remain unchanged. It is helpful to look now in more detail at the final digits of the fourth powers which are tabulated here for clarity:

Digit	1	2	3	4	5	6	7	8	9
Final digit of the fourth power	1	6	1	6	5	6	1	6	1

The odd numbers, apart from 5 are being multiplied by unity and so they obviously remain unchanged. 5 is multiplied by 5 and likewise the final digit remains unchanged as 5. But why does the final digit of the even numbers remain unchanged when multiplied by 6? E.g. 2 x 6 = 12, which ends in 2, 4x 6 = 24 which end in 4 and similarly for 6 and 8.

The solution is easy to see when we partition the 6 into (5 + 1) and consider the final digit only. So 4 x 6 = 4 x (5 + 1) and in general, since we are considering the even digits only, we can write the general statement: 2a x (5+1) where 2a is the original even number.

Since 2a x (5 +1) = 10a + 2a, it is only the 2a, the original digit, which remains in the units position since 10a gives a digit in the tens column. Thus 6 is the " identity element" for the last digit for the even digits 2, 4, 6 and 8 and so these digits have the same final digit in the product when multiplied by 6.

On this reasoning, we would expect, 2^{nd}, 6^{th}, 10^{th} etc powers of the 9 digits to have their final digits unchanged and this is indeed the case, and so on for other powers whose common ratio is the 4^{th} power.

Finally, it will be noticed in table 3 that the last digits of the even powers are not unique as they are in the case of odd powers and so the method explained to extract cube roots, 5^{th} roots etc above cannot be used to extract the roots of even powers without adjustment. For example the squares of the nine digits are 1, 4, 9, 16, 25, 36, 49, 64, 81 whose last digits are respectively 1,4,9,6,5,6,9,4,1. Apart from 5, the final digit of the square is not unique. However, by continually extracting the square root of the even power until an odd power is achieved the method can be adapted to include even powers. For instance, to find the sixth root of a number, first extract the square root and then find the cube root of the resulting number.

e.g. To find the sixth root of 2 565 726 409 first extract the square root by the Dwandwa yoga method (See Chapter 34 of "Vedic Mathematics") giving 50 653 and then extract the cube root by the method explained earlier, giving 37 as the required answer.

Thus it will be seen that there are comprehensive laws governing the beauty of the patterns observed in the powers and their roots, which, if fully understood, enable anyone to extract the roots of any exact power of any two digit number.

Not only that, it is clear that by using the appropriate Vedic sutras, seemingly difficult and complex calculations such as extracting roots of any powers can be performed very simply and indeed entirely mentally which are marking features of Vedic Mathematics.

R.McNeill   April 15