Trigonometry: The Vedic Way

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Abstract

Trigonometric identities are an integral part of trigonometry. In this paper, we aim to prove various trigonometric identities using Vedic Math methods, primarily 'Triples and the sutra 'By mere observation'. Using these Vedic Math techniques, we can arrive at a simpler and, compared with conventional derivations, more intuitive proofs for these identities. These help us to reinforce the concepts in our minds.

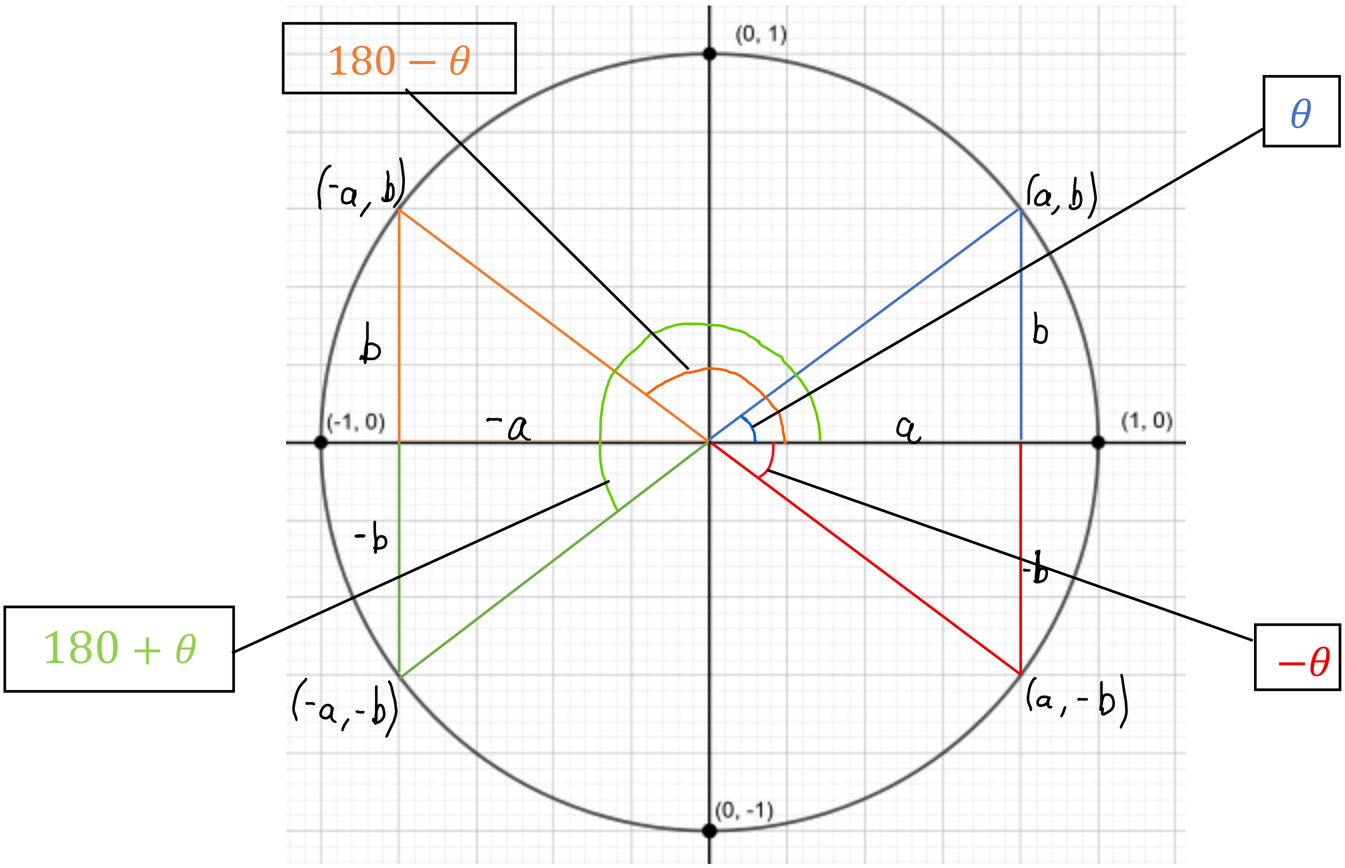
Objectives

The primary objectives of this paper are:

- To apply Vedic Mathematics techniques to the derivation of trigonometric identities.
- To leverage the unit circle method for quadrant-based derivations.
- To implement the triples method for formula simplification.
- To demonstrate how Vedic Math sutras such as "*Vilokanam*" (By Mere Observation) facilitate quicker calculations.
- To compare the efficiency of Vedic-based derivations with conventional methods.

Quadrant Angles

To find the values of trigonometric ratios of angles in the cartesian plane, there are certain conventions about the signs of each of the ratios. To understand these, we make use of a unit circle (a circle with radius 1 unit). Here, the angle θ is an acute angle.



In the above figure, (a,b) refers to a point on the cartesian plane. Thus,

$$a = cos\theta$$

$$b = sin\theta$$

This applies for θ, $-\theta$, $180-\theta$, and $180+\theta$. In the graph, a and b have different signs in different quadrants. Thus, we can find out the sign of the trigonometric ratio by considering the sign of the coordinates in that quadrant. The sign of the abscissa can be used to figure out the sign of the cosine of the angle and the sign of the ordinate can be used to figure out the sign of the sine of the angle. Thus,

- In the first quadrant (for θ), a and b are positive. This means that both $cos\,\theta$ and $sin\,\theta$ are positive.

- In the second quadrant (for $180 - \theta$), a is negative and b is positive. This means that $cos\,\theta$ is negative and $sin\,\theta$ is positive.
- In the third quadrant (for $180 + \theta$), both a and b are negative. This means that both $cos\,\theta$ and $sin\,\theta$ are negative.
- In the fourth quadrant (for $-\theta$), a is positive and b is negative. This means that $cos\,\theta$ is positive and $sin\,\theta$ is negative.

By Mere Observation

Now, we can find the values of various trigonometric ratios without much calculation. There are certain things to keep in mind:

- If we have to find a ratio for θ or $(180 \pm \theta)$, we retain the triple for the reference angle (θ) and change the sign of the output according to the quadrant.
  Eg: To find $cos210°$,
  $$210° = 180° + 30°$$
  Thus, $30°$ is our refence angle. So, the triple will be:
  $$30°)\ \sqrt{3} \quad 1 \quad 2$$
  Since, we're adding it to $180°$, the triple remains the same. We only need to change the sign of the adj and opp side values. So, the triple will now be:
  $$210°) -\sqrt{3} \quad -1 \quad 2$$
  Thus, $cos210° = -\frac{\sqrt{3}}{2}$.

- If we have to find a ratio for $90 \pm \theta$ or $(270 \pm \theta)$, we switch the adj and opp side values for the triple of the reference angle (θ) and change the sign of the output according to the quadrant.
  Eg: To find $tan240°$,
  $$240° = 270° - 30°$$
  Thus, $30°$ is our refence angle. So, the triple will be:
  $$30°)\ \sqrt{3} \quad 1 \quad 2$$

Since, we're subtracting it from $270°$, the triple gets inversed. Now, we also need to change the sign of the adj and opp side values. So, the triple will now be:

$$240°) -1 \quad -\sqrt{3} \quad 2$$

Thus, $tan240° = \frac{-\sqrt{3}}{-1} = \sqrt{3}$.

This entire process can easily be performed mentally by *mere observation*.

Using Pythagoras Theorem

Directly using the Pythagoras theorem, we can derive a few more important trigonometric identities. The Pythagoras theorem can be represented as:

$$a^2 + b^2 = c^2$$

Or, we may rewrite it in terms of the sides with respect to an angle, like so:

$$adj^2 + opp^2 = hyp^2$$

Where adj = adjacent side, opp = opposite side, and hyp = hypotenuse. Now, to derive the first identity, we can divide the whole equation by the square of the hypotenuse (hyp^2):

$$\frac{adj^2}{hyp^2} + \frac{opp^2}{hyp^2} = \frac{hyp^2}{hyp^2}$$

$$\mathbf{\cos^2 \theta + \sin^2 \theta = 1}$$

Similarly, when we divide the entire equation by the square of the adjacent side (adj^2), we get:

$$\frac{adj^2}{adj^2} + \frac{opp^2}{adj^2} = \frac{hyp^2}{adj^2}$$

$$\mathbf{1 + \tan^2 \theta = \sec^2 \theta}$$

Rewriting this equation, we also get:

$$\mathbf{\sec^2 \theta - \tan^2 \theta = 1}$$

$$\mathbf{\sec^2 \theta - 1 = \tan^2 \theta}$$

And, when we divide the entire equation by the square of the opposite side (opp^2), we can derive:

$$\frac{adj^2}{opp^2} + \frac{opp^2}{opp^2} = \frac{hyp^2}{opp^2}$$

$$\mathbf{cot^2\,\boldsymbol{\theta} + 1 = cosect^2\,\boldsymbol{\theta}}$$

And rewriting this equation, we get:

$$\mathbf{cosec^2\,\boldsymbol{\theta} - \boldsymbol{cot^2\theta} = 1}$$

$$\mathbf{cosec^2\,\boldsymbol{\theta} - 1 = cot^2\,\boldsymbol{\theta}}$$

Trigonometric Ratios for the Sum and Difference of Angles

Let's take a right triangle with hypotenuse of one unit. Then the sides would be $cosA$ and $sinA$. Similarly, taking another right triangle with hypotenuse of one unit, then the sides would be $cosB$ and $sinB$. Now we add the triples using triple addition:

A)	$\cos A$	$\sin A$	1
B)	$\cos B$	$\sin B$	1
A+B)	$\cos A \cos B -$ $\sin A \sin B$	$\sin A \cos B +$ $\cos A \sin B$	1

Thus, we have obtained three of our identities:

$$\mathbf{sin(\boldsymbol{A+B}) = \boldsymbol{sinAcosB + cosAsinB}}$$

$$\mathbf{cos(\boldsymbol{A+B}) = \boldsymbol{cosAcosB - sinAsinB}}$$

$$\tan(A+B) = \frac{sinAcosB + cosAsinB}{cosAcosB - sinAsinB}$$

By dividing the Numerator and Denominator by $\boldsymbol{cosAcosB}$

$$\frac{\frac{sinAcosB + cosAsinB}{cosAcosB}}{\frac{cosAcosB - sinAsinB}{cosAcosB}}$$

Thus,

$$\mathbf{tan}\,(\boldsymbol{A}+\boldsymbol{B}) = \frac{\boldsymbol{tanA}+\boldsymbol{tanB}}{\mathbf{1}-\boldsymbol{tanAtanB}}$$

Now, we do triple subtraction to get our next three identities:

A) $\cos A$	$\sin A$	1
B) $\cos B$	$\sin B$	1
A-B) $\cos A \cos B + \sin A \sin B$	$\sin A \cos B - \cos A \sin B$	1

Thus, we have directly proven three more identities:

$$\mathbf{sin}(\boldsymbol{A}-\boldsymbol{B}) = \boldsymbol{sinAcosB} - \boldsymbol{cosAsinB}$$

$$\mathbf{cos}(\boldsymbol{A}-\boldsymbol{B}) = \boldsymbol{cosAcosB} + \boldsymbol{sinAsinB}$$

$$\tan(A-B) = \frac{sinAcosB - cosAsinB}{cosAcosB + sinAsinB}$$

By dividing the Numerator and Denominator by $cosAcosB$:

$$\frac{\frac{sinAcosB - cosAsinB}{cosAcosB}}{\frac{cosAcosB + sinAsinB}{cosAcosB}}$$

Thus,

$$\mathbf{tan}\ (\boldsymbol{A}-\boldsymbol{B})=\frac{\boldsymbol{tanA}-\boldsymbol{tanB}}{\mathbf{1}+\boldsymbol{tanAtanB}}$$

Double Angle

A) $\cos A$	$\sin A$	1
2A)$\cos^2 A-\sin^2 A$	$2\sin A\cos A$	1

$$\mathbf{sin2A}=\mathbf{2sinAcosA}$$

$$\boldsymbol{cos2A}=\mathbf{cos}^{\mathbf{2}}\boldsymbol{A}-\ \mathbf{sin}^{\mathbf{2}}\boldsymbol{A}$$

$$=\mathbf{1}-\mathbf{2}\,\mathbf{sin}^{\mathbf{2}}\boldsymbol{A}$$

$$=\mathbf{2}\,\mathbf{cos}^{\mathbf{2}}\boldsymbol{A}-\mathbf{1}$$

$$tan2A=\ \frac{2sinAcosA}{\cos^2 A-\sin^2 A}$$

Dividing the numerator and denominator by $\cos^2 \mathrm{A}$, we get:

$$\boldsymbol{tan2A}=\frac{\boldsymbol{2tanA}}{\mathbf{1}-\mathbf{tan}^{\mathbf{2}}\boldsymbol{A}}$$

Triple Angle

A) $\cos A$	$\sin A$	1
2A)$\cos^2 A-\sin^2 A$	$2\sin A\, cosA$	1
3A) $cosA(\cos^2 A-\sin^2 A)-$ $sinA(2sinAcosA)$	$\sin A(\cos^2 A-\sin^2 A)$ $+cosA(2sinAcosA)$	1

Thus, we have derived the following:

$$sin3A = sinA(\cos^2 A - \sin^2 A) + cosA(2sinAcosA)$$

$$= sinA\cos^2 A - \sin^3 A + 2sinA\cos^2 A$$

$$= 3sinA\cos^2 A - \sin^3 A$$

$$= (3sinA)(1 - \sin^2 A) - \sin^3 A$$

$$= 3sinA - 3\sin^3 A - \sin^3 A$$

$$= 3sinA - 4\sin^3 A$$

$$\boldsymbol{sin3A = 3sinA - 4\sin^3 A}$$

$$cos3A = cosA(\cos^2 A - \sin^2 A) - sinA(2sinAcosA)$$

$$= \cos^3 A - cosA\sin^2 A - 2cosA\sin^2 A$$

$$= \cos^3 A - 3cosA\sin^2 A$$

$$= \cos^3 A - 3cosA(1 - \cos^2 A)$$

$$= \cos^3 A - 3cosA + 3\cos^3 A$$

$$= 4\cos^3 A - 3cosA$$

$$\boldsymbol{cos3A = 4\cos^3 A - 3cosA}$$

$$tan3A = \frac{sinA(\cos^2 A - \sin^2 A) + cosA(2sinAcosA)}{cosA(\cos^2 A - \sin^2 A) - sinA(2sinAcosA)}$$

$$= \frac{sinA(\cos^2 A - \sin^2 A + 2\cos^2 A)}{cosA(\cos^2 A - \sin^2 A - 2\sin^2 A)}$$

$$= tanA\frac{(3\cos^2 A - \sin^2 A)}{(\cos^2 A - 3\sin^2 A)}$$

Divide the Numerator and Denominator by $cos^2 A$:

$$tanA \frac{\frac{(3\cos^2 A - \sin^2 A)}{\cos^2 A}}{\frac{(\cos^2 A - 3\sin^2 A)}{\cos^2 A}}$$

$$= tanA \frac{(3 - \tan^2 A)}{(1 - 3\tan^2 A)}$$

$$= \frac{3tanA - \tan^3 A}{1 - 3\tan^2 A}$$

Thus,

$$\boldsymbol{tan3A = \frac{3tanA - \tan^3 A}{1 - 3\tan^2 A}}$$

Sum and Difference of Trigonometric Ratios

Now, we know that:

$$\sin(A + B) = sinAcosB + cosAsinB$$
$$\cos(A + B) = cosAcosB - sinAsinB$$
$$\sin(A - B) = sinAcosB - cosAsinB$$
$$\cos(A - B) = cosAcosB + sinAsinB$$

Thus,

$$sin(A + B) + sin(A - B) = sinAcosB + cosAsinB + sinAcosB - cosAsinB$$
$$= 2sinAcosB$$

Substitute $(A + B)$ with C and $(A - B)$ with D

$$sinC + sinD = 2\sin \frac{2A}{2} cos\frac{2B}{2}$$

$$sinC + sinD = 2\sin \frac{[(A+B)+(A-B)]}{2} cos\frac{[(A+B)-(A-B)]}{2}$$

Thus,

$$\boldsymbol{sinC + sinD = 2\sin \frac{C+D}{2} cos\frac{C-D}{2}}$$

Similarly,

$$sin(A+B) - sin(A-B) = \ sinAcosB + cosAsinB - (sinAcosB - cosAsinB)$$

$$sin(A+B) - sin(A-B) = 2cosAsinB$$

Substituting $(A+B)$ with C and $(A-B)$ with D, we get:

$$sinC - sinD = 2\cos\frac{2A}{2} sin\frac{2B}{2}$$

$$sinC - sinD = 2\cos \frac{[(A+B)+(A-B)]}{2} sin\frac{[(A+B)-(A-B)]}{2}$$

Thus,

$$\boldsymbol{sinC - sinD = 2\cos \frac{C+D}{2} sin\frac{C-D}{2}}$$

Next,

$$\cos(A+B) + \cos\,(A-B)$$

$$= cosAcosB - sinAsinB + cosAcosB + sinAsinB$$

$$= 2cosAcosB$$

Now, replace $(A+B)$ with C and $(A-B)$ with D:

$$2cos\frac{2A}{2}cos\frac{2B}{2}$$

$$= 2cos\frac{(A+B)+(A-B)}{2}cos\frac{(A+B)-(A-B)}{2}$$

Thus,

$$\boldsymbol{cosC + cosD = 2cos\frac{C+D}{2}\cos\frac{C-D}{2}}$$

And, finally for the last identity:

$$\cos(A+B) - \cos(A-B)$$

$$= (cosAcosB - sinAsinB) - (cosAcosB + sinAsinB)$$

$$= -2sinAsinB$$

Now replace$(A+B)$ with C and $(A-B)$ with D:

$$-2sin\frac{2A}{2}sin\frac{2B}{2}$$

$$= -2sin\frac{(A+B)+(A-B)}{2}sin\frac{(A+B)-(A-B)}{2}$$

Thus, we have:

$$\boldsymbol{cosC - cosD = -2sin\frac{C+D}{2}\sin\frac{C-D}{2}}$$

Comparison

Using these Vedic Math techniques, we can arrive at solutions to problems in a faster and simpler way, as compared with the traditional way of solving. For example, to calculate the trigonometric ratios of angles greater than 90°, we can easily make use of the unit circle method or the mere observation method to get our answer quickly.

Eg: Find $\sin(330°)$

Ans 1 – Using Traditional Method

$$sin330° = \sin(270 + 60)°$$

Now, using the formula,

$$\sin(A + B) = sinA\ cosB + cosA\ sinB$$

We get:

$$sin330° = sin270\ cos60 + sin60\ cos270$$

$$= (-1)\left(\frac{1}{2}\right) + \left(\frac{\sqrt{3}}{2}\right)(0)$$

$$= -\frac{1}{2}$$

Ans 2 – Using Vedic Math Method

$$330° = 270° + 60°$$

So, our triple is $60°)\ \ 1\ \ \sqrt{3}\ \ \ 2$

Thus, $330°)\sqrt{3}\ \ -1\ \ \ 2$

And we have our final answer:

$$sin330° = -\frac{1}{2}$$

To find the trigonometric ratios of certain non-standard angles, we can use the triple addition or subtraction methods.

Eg: Find $cos15°$

Answer 1 – Traditional Method

$$15° = 45° - 30°$$

Thus, we can use the formula,

$$cos(A - B) = cosA\ cosB + sinA\ sinB$$

So, we get

$$\cos 15° = \cos 45° \cos 30° + \sin 45° \sin 30°$$

$$= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \cdot \frac{1}{2}$$

$$= \frac{\sqrt{3}+1}{2\sqrt{2}}$$

Answer 2 – Vedic Math Method

$45°)\ 1$	1	$\sqrt{2}$
$30°)\ \sqrt{3}$	1	2
$15°)\sqrt{3}+1$	—	$2\sqrt{2}$

Thus, we directly procured our final answer, without having to use any formulae.

$$\cos 15° = \frac{\sqrt{3}+1}{2\sqrt{2}}$$

Conclusion

By integrating Vedic Mathematics techniques into trigonometric proofs, we provide a fresh perspective that enhances both speed and simplicity. The unit circle method and the triples approach eliminate redundant calculations and the need to memorize countless formulae, offering a more intuitive path to understanding trigonometric identities. While traditional proofs remain foundational, Vedic Math techniques serve as effective alternatives that can accelerate problem-solving without the need for formulae.

References

1. Triples – By Prof. Kenneth R. Williams
2. Mathematics Textbook for Class XI – NCERT